

Fast methods to compute the Riemann zeta function

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Abstract

The Riemann zeta function on the critical line can be computed using a straightforward application of the Riemann-Siegel formula, Schönhage's method, or Heath-Brown's method. The complexities of these methods have exponents $1/2$, $3/8$ ($=0.375$), and $1/3$ respectively. In this paper, three new fast and potentially practical methods to compute zeta are presented. One method is very simple. Its complexity has exponent $2/5$. A second method relies on this author's algorithm to compute quadratic exponential sums. Its complexity has exponent $1/3$. The third method employs an algorithm, developed in this paper, to compute cubic exponential sums. Its complexity has exponent $4/13$ (approximately, 0.307).

1 Introduction

The Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1$$

It can be continued analytically to the entire complex plane except for a simple pole at $s = 1$. The values of $\zeta(1/2 + it)$ on finite intervals are of interest to number-theorists. For example, researchers investigating moment questions or more recently links to Random Matrix Theory models often utilize numerical evidence about the zeta function on the critical line. Also, the numerical verification of the Riemann hypothesis is clearly dependent on this kind of data. Such considerations led to pursuits of efficient methods to *compute* $\zeta(1/2 + it)$ *with polynomial accuracy*; that is, methods to numerically evaluate $\zeta(1/2 + it)$ for $t > t_0$, t_0 a fixed positive number, with absolute error bounded by $t^{-\lambda}$ for fixed positive λ . Searches for such methods were also motivated from a computational complexity perspective; namely, the zeta function is of fundamental importance in number theory, so, one may simply ask, how fast can it be computed?

The aim of this paper is to present new efficient methods to compute zeta. In particular, our fastest method, which has complexity $t^{4/13+o(1)}$ ($4/13 \approx 0.307$),

improves on the best previously known $t^{1/3+o(1)}$ “complexity bound” to compute zeta by a noticeable margin.

Early examples of methods to compute zeta include the Euler-Maclaurin formula (see [O]) and the Riemann-Siegel formula (see [T]). The problem was also tackled by Odlyzko [OS] [O], Schönhage [OS] [S], Heath-Brown [HB], Turing [Tu], Berry and Keating [BK], and Rubinstein [R], among others.

Most methods to compute the zeta function rely on the Riemann-Siegel formula as starting point. One frequently used version of the formula is

$$\zeta(1/2 + it) = \Re \left(2e^{i\theta(t)} \sum_{n=1}^{n_1} n^{-1/2} \exp(it \log n) \right) + \Phi(t) + R(t) \quad (1)$$

where $n_1 := \lfloor \sqrt{t/(2\pi)} \rfloor$, $\theta(t)$ and $\Phi(t)$ are certain real functions that can be evaluated in $O(\log t)$ time, and the error function $R(t)$ satisfies

$$|R(t)| \leq 0.053t^{-5/4} \quad \text{for } t \geq 200 \quad (2)$$

See [E]. The bound (2) suffices for most applications in the literature. There exist other more accurate versions of the Riemann-Siegel formula; that is, versions with a different $\Phi(t)$ and an error function $R(t)$ that declines faster than $t^{-5/4}$. In this paper, we only attack the problem of computing the main sum in (1) with polynomial accuracy. In this sense, our zeta methods are essentially independent of the precise choice of the functions $\Phi(t)$ and $R(t)$.

Given the computational resources of today, one can easily use the Riemann-Siegel formula to compute $\zeta(1/2 + it)$ at a *single* point for t around 10^{25} . However, due to probabilistic or asymptotic considerations, one is often interested in the values of zeta at a relatively large number of points; consider the computations of Odlyzko [O] and Odlyzko and Hiary [HO] for example. The amortized complexity methods of Odlyzko and Schönhage [OS] are designed to satisfy exactly this requirement. Specifically, their methods permit the evaluation of $\zeta(1/2 + iT + it)$ at $O(T^{1/2})$ points in the interval $t \in [0, T^{1/2}]$ with complexity $T^{1/2+o(1)}$.

Despite the distinct advantages offered by the Odlyzko-Schönhage algorithms, their methods did not improve the complexity of computing the zeta function at a single point. A single evaluation of $\zeta(1/2 + it)$ still consumed $O(t^{1/2})$. Schönhage [S] used the Fast Fourier Transform (FFT) and subdivisions of the main sum in the Riemann-Siegel formula (see Section 2) to reduce the cost of a single evaluation of $\zeta(1/2 + it)$ to $t^{3/8+o(1)}$. Later, Heath-Brown [HB] presented ideas that could improve the complexity of a single evaluation to $t^{1/3+o(1)}$. He described his approach in the following way:

The underlying idea was to break the zeta-sum into $t^{1/3}$ subsums of length $t^{1/6}$, on each of which $\exp(it \log(n+h))$ could be approximated by a quadratic exponential $e(Ah + Bh^2 + f(h))$ with $f(h) = O(1)$. One would then pick a rational approximation a/q to B and write the sum in terms of complete Gauss sums to modulus q .

This is motivated by section 5.2 in Titchmarsh, but using explicit formulae with Gauss sums in place of Weyl squaring.

The work of [S], [H1], and [HB] (see also [T], page 99) makes it apparent that, a possible approach to improving the complexity of computing zeta is to derive efficient methods to numerically evaluate sums of the form

$$\frac{1}{K^j} \sum_{k=K_1}^{K_2} k^j \exp(2\pi i f(k)), \quad f(x) \in \mathbb{R}[x] \quad (3)$$

This is our approach to improving the complexity of computing zeta.

The paper is organized as follows. In Section 2, we describe the relation between sums of the form (3) and the zeta function. In particular, we derive methods to compute $\zeta(1/2 + it)$ with complexities $t^{2/5+o(1)}$, $t^{1/3+o(1)}$, and $t^{4/13+o(1)}$. The first method is very simple. The second and third methods rely on the existence of “efficient” algorithms to compute “quadratic” exponential sums and “cubic” exponential sums with a relatively small cubic coefficient. By “quadratic” and “cubic” we mean sums of the form (3) with $f(x)$ a polynomial of degree 2 and 3 respectively. A nearly-optimal algorithm to compute quadratic sums has already been derived in [H1]. As for cubic sums, they can be evaluated using the algorithm presented in Sections 3, 4, and 5 of this paper. Specifically, Section 3 contains an informal description of our algorithm to compute cubic sums, Section 4 details the steps involved in the algorithm, and Section 5 contains proofs of various lemmas employed in Section 4.

We wish to make a few comments about the structure and presentation of our algorithm to compute cubic sums. This algorithm is the essential component of our $t^{4/13+o(1)}$ method to compute zeta.

Apart from the use of FFT and a few other ideas, the cubic sums algorithm is essentially a generalization of the quadratic sums method of [H1]. Therefore, it is helpful to familiarize one’s self with the ideas of [H1] first. Also, the goal of this paper is to compute cubic sums rather than derive elegant asymptotic expressions for them. Thus, for example, in Section 4 we explain that the cubic algorithm involves evaluating certain exponential integrals. In Section 5, we show how to compute each of these integrals separately. By doing so, we obscure the fact it is possible to combine some of these integrals in a manner that might collapse them. We do not concern ourselves with simplifying such integrals as this does not improve the exponent of the complexity.

To summarize, Sections 4 and 5 communicate the rigorous details of our cubic sums algorithm. If one merely wishes to acquire a taste for its general structure without getting too involved in the details, then reading Section 3 will likely suffice.

2 Outline of fast methods to compute $\zeta(1/2 + it)$

We show how to compute $\zeta(1/2 + it)$ with polynomial accuracy. We borrow some of Schönhage’s [S] notation. Assume $t > 10^6$ and let $\beta \in [0, 1/4]$. We

mention from the outset that to obtain our $t^{2/5+o(1)}$, $t^{1/3+o(1)}$, and $t^{4/13+o(1)}$ complexities, the parameter β has to be specialized to $1/10$, $1/6$, and $5/26$ respectively. Also, the complexities obtained in this section (i.e. $2/5$, $1/3$, and $4/13$) are the outcomes of simple optimization procedures that we do not make explicit. Let us specify our computational model and some notational issues first.

An *arithmetic operation* means an addition, a multiplication, an evaluation of the logarithm of a positive number, or an evaluation of complex exponential. We perform arithmetic using $O((\log t)^2)$ bits. We remark that in practical versions of our algorithms, many fewer bits will suffice to perform arithmetic. We measure computational complexity (or time) by the number of arithmetic operations required. This in turn can be routinely bounded in terms of bit operations because all the numbers that occur in our algorithm have $O((\log t)^2)$ bits. Frequently, we abbreviate “arithmetic operations” to simply “operations.”

Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x , $\lceil x \rceil$ denote smallest integer greater than or equal to x , $\{x\}$ denote $x - \lfloor x \rfloor$, and $\log x$ denotes $\log_e x$. Let $\exp(x)$ and e^x both stand for the usual exponential function (they are used interchangeably). We define $0^0 := 1$ whenever it occurs. Finally, all big- O notation constants are absolute.

We now describe our methods to compute zeta with error bounded by $t^{-\lambda+o(1)}$, λ fixed positive constant. Subdivide the Riemann-Siegel sum (1) into $O(\log t)$ subsums of the form

$$\sum_{n=N}^{2N} n^{-1/2} \exp(it \log n), \quad (4)$$

where $t^{1/2-\beta} \leq N \leq \sqrt{t/(2\pi)}$, plus a remainder Dirichlet series of length $O(t^{1/2-\beta})$. We evaluate the remainder series directly. Choose $K \approx Nt^{\beta-1/2}$, say K satisfies $Nt^{\beta-1/2} < K \leq Nt^{\beta-1/2} + 1$. Rewrite (4) as

$$\sum_{v \in V} \sum_{k=0}^K \frac{\exp(it \log(v+k))}{\sqrt{v+k}} + \text{Remainder} \quad (5)$$

where V is a set of $\lfloor N/K \rfloor$ equally spaced points in the interval $[N, 2N]$ and *Remainder* is a Dirichlet series of length $O(K) = O(t^\beta)$, which we evaluate directly. The main sum in (5) can be rewritten as

$$\sum_{v \in V} \frac{\exp(it \log v)}{\sqrt{v}} \sum_{k=0}^K \frac{\exp(it \log(1+k/v))}{\sqrt{1+k/v}} \quad (6)$$

Using Taylor expansions, the inner sums in (6) can be expressed in the form

$$\sum_{l=0}^{\infty} \frac{(-1)^l \prod_{s=1}^l (2s-1)}{l! 2^l v^l} \sum_{k=0}^K k^l \exp \left(it \sum_{m=1}^{\infty} \frac{(-1)^{m+1} k^m}{m v^m} \right) \quad (7)$$

Since $k/v \leq K/N \leq t^{\beta-1/2}$, we can truncate both Taylor series in (7) to obtain

$$\sum_{l=0}^L \frac{(-1)^l \prod_{s=1}^l (2s-1)}{l! 2^l v^l} \sum_{k=0}^K k^l \exp \left(it \sum_{m=1}^M \frac{(-1)^{m+1} k^m}{m v^m} + \epsilon_{k,M} \right) + \delta_L \quad (8)$$

where

$$\begin{aligned} \epsilon_{k,M} &= O(t K^M / N^M) = O(t^{1-(1/2-\beta)M}) \\ \delta_L &= O(K^{L+1} / N^L) = O(t^{\beta-(1/2-\beta)L}) \end{aligned}$$

Choose $L = M = \lceil (\lambda + 2)/(1/2 - \beta) \rceil$. Note that $M = O(1)$. By simple calculations, the sum (8) is equal to

$$\sum_{l=0}^M \frac{(-1)^l \prod_{s=1}^l (2s-1)}{l! 2^l v^l} \sum_{k=0}^K k^l \exp \left(it \sum_{m=1}^M \frac{(-1)^{m+1} k^m}{m v^m} \right) + O(t^{-\lambda-1}) \quad (9)$$

The inner sums in (9) can be reduced to the form

$$\frac{1}{K^l} \sum_{k=0}^K k^l \exp \left(it \sum_{m=0}^{m'-1} \frac{(-1)^{m+1} k^m}{m v^m} + it \sum_{m=m'}^M \frac{(-1)^{m+1} k^m}{m v^m} \right) \quad (10)$$

where $0 \leq l \leq M$ and $1 \leq m' \leq M$ are integers. Let $m' = \lceil 1/(1/2 - \beta) \rceil$. Then the sum (10) can be reformulated as

$$\frac{1}{K^l} \sum_{k=0}^K k^l \exp \left(it \sum_{m=0}^{m'-1} \frac{(-1)^{m+1} k^m}{m v^m} \right) \left(\sum_{s=0}^{\infty} \frac{c_k^s (-1)^{s(m'+1)} (it)^s k^{sm'}}{s! (m')^s v^{sm'}} \right) \quad (11)$$

where

$$c_k = \sum_{m=0}^{M-m'} \frac{(-1)^m m' k^m}{(m+m') v^m}$$

Note that $c_k = 1 + \delta$, where $|\delta| \leq 2k/v \leq 2t^{\beta-1/2}$. We can truncate the Taylor series with respect to s in (11) after $R = \lceil (\lambda + 2) \log t \rceil$ terms. We obtain

$$\frac{1}{K^l} \sum_{k=0}^K k^l \exp \left(it \sum_{m=0}^{m'-1} \frac{(-1)^{m+1} k^m}{m v^m} \right) \left(\sum_{s=0}^R \frac{c_k^s (-1)^{s(m'+1)} (it)^s k^{sm'}}{s! (m')^s v^{sm'}} \right) + O(t^{-\lambda-1}) \quad (12)$$

The sums (12) can be easily reduced to the form

$$\frac{1}{K^j} \sum_{k=0}^K k^j \exp \left(\sum_{m=0}^{m'-1} a_m k^m \right), \quad a_m \in [-t/N^m, t/N^m], \quad (13)$$

where $0 \leq j = O(\log t)$ is an integer. Finally, suppose any sum of the form (13) can be evaluated with error $O(t^{-\lambda-1}) = O(K^{-(\lambda+1)/\beta})$ in $t^{o(1)}$ time, where we

allow for a precomputation costing $t^{1/2-\beta+o(1)}$ operations. Then $\zeta(1/2+it)$ can be computed in $t^{1/2-\beta+o(1)}$ time with error bounded by $t^{-\lambda+o(1)}$.

We specialize β to obtain the complexities stipulated at the beginning of the section. First, to compute zeta in $t^{2/5+o(1)}$ operations, choose $\beta = 1/10$. Then by the previous argument, we only need to be able to evaluate sums of the form

$$F(K, j; a, b) = \frac{1}{K^j} \sum_{k=0}^K k^j \exp(2\pi i a k + 2\pi i b k^2),$$

$$a, b \in \mathbb{R}/\mathbb{Z}, \quad 0 \leq j = O(\log t) \quad (14)$$

in $t^{o(1)}$ time, where we allow for a precomputation costing $t^{2/5+o(1)}$ operations. To accomplish this, we can directly precompute the functions

$$F(K, l; \tilde{a}, \tilde{b}), \quad 0 \leq l = O(\log t) \quad (15)$$

at all points (\tilde{a}, \tilde{b}) in the set

$$L := \{(\delta/K, \tau/K^2) | 0 \leq \delta < K, 0 \leq \tau < K^2\}$$

The cost of the precomputation is $K^{4+o(1)} = t^{2/5+o(1)}$. Once the precomputation is carried out, Taylor expansions can be used to evaluate $F(K, j; a, b)$ for any $(a, b) \in [0, 1) \times [0, 1)$ with error bounded by $t^{-\lambda+o(1)}$. Explicitly, given a pair (a, b) which is not necessarily a member of the lattice L , let (\tilde{a}, \tilde{b}) a member of L closest to (a, b) . Also, let $\Delta a := K(a - \tilde{a})$ and $\Delta b := K^2(b - \tilde{b})$. Then,

$$F(K, j; a, b) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l=0}^n \binom{n}{l} F(K, j + 2n - l; \tilde{a}, \tilde{b}) (\Delta a)^l (\Delta b)^m \quad (16)$$

It is easily seen that the series indexed by n in (16) can be truncated after $O(\log t)$ terms.

Our $t^{2/5+o(1)}$ method is simple to program. The only drawback is the data resulting from the precomputation has to be temporarily stored. Considering the range of t where the method can be realistically applied, and given today's computational resources, the amount of data to be stored is certainly manageable. Let us give an example.

Suppose we want to compute zeta when $t \approx 10^{20}$ with absolute error less than 10^{-6} . This is a medium level of accuracy, but it is generally sufficient to perform moment calculations and separate zeroes for instance. Since we are discussing matters of practicality, let us assume errors in our numerical evaluations of the inner sums in (6) behave as independent random variables of mean zero. This assumption is not needed for the theoretical complexity of our method, but it a reasonable and generally reliable one to make, see [O] for example. With this setup, one finds via quick accounting that it is enough to take $0 \leq l \leq 11$ in (15) and store the precomputed values of $F(K, l; \tilde{a}, \tilde{b})$ using 32-bit numbers. So, the amount of data to be stored is $\approx (8/7) \times 10^6 \times 12 \times 4$ bytes, which is about 0.05 Gigabytes. Note however, in order to ensure there is

a significant speed-up resulting from the use of our method, one would need to precompute $F(K, l; \tilde{a}, \tilde{b})$ on a denser lattice than L . Since a denser grid implies fewer derivatives of $F(K, l; \tilde{a}, \tilde{b})$ would be needed, the functions $F(K, j; a, b)$ can then be evaluated more quickly. For example, if we use a lattice that is 17 times denser, we would need to store about 9 Gigabytes of data instead. The larger 9 Gigabyte dataset is still of moderate size. Moreover, it would allow for a speed-up by a factor of 5 to 10 compared to the Riemann-Siegel formula. This is a respectable improvement considering the elementary nature of the method.

To summarize, although our $t^{2/5+o(1)}$ method is not as fast as those of Schönhage and Heath-Brown, it is far easier to implement and its error terms are generally easier to control. It is a potential replacement for the straightforward application of the Riemann-Siegel formula for t greater than 10^{20} , and also for lower ranges of t provided the implementation is reasonably optimized.

To derive our $t^{1/3+o(1)}$ zeta method, choose $\beta = 1/6$. We need to be able to compute sums of the form $F(K, j; a, b)$ in $t^{o(1)}$ operations but this time the upper bound for the cost of our precomputation is only $t^{1/3+o(1)}$ operations. Recently, this author [H1] has derived a polynomial-time algorithm to compute $F(K, j; a, b)$ as well as sums of the form

$$G(K, j; a, b) := \sum_{k=1}^K \frac{1}{k^j} \exp(2\pi i a k + 2\pi i b k^2), \quad a, b \in \mathbb{R}/\mathbb{Z}, \quad 0 \leq j = O(\log t)$$

Specifically, it was proved

Theorem 2.1. *There exist absolute constants κ_1 and κ_2 such that for any $\epsilon \in (0, e^{-1})$, any positive integer K , any non-negative integer j , any $a, b \in \mathbb{R}/\mathbb{Z}$, and with an underlying computational model that performs arithmetic operations using $O(\nu^2)$ -bit arithmetic, where $\nu = \nu(K, j, \epsilon) := (j+1) \log(K/\epsilon)$, each of the functions $F(K, j; a, b)$ and $G(K, j; a, b)$ can be computed with absolute error bounded by $O(\nu^{\kappa_1} \epsilon)$ using $O(\nu^{\kappa_2})$ arithmetic operations. The big- O constants are absolute.*

By choosing ϵ to be some fixed negative power of K , we obtain a method to compute zeta in $t^{1/3+o(1)}$ operations. There are two important practical advantages to this method. First, it does not require performing any precomputations or storing any data. Second, it is not too hard to implement.

In Section 4, we take Theorem 2.1 a step further. We tackle the sums

$$H(K, j; a, b, c) := \frac{1}{K^j} \sum_{k=0}^K k^j \exp(2\pi i a k + 2\pi i b k^2 + 2\pi i c k^3)$$

when c is relatively small. Our goal is to derive an efficient algorithm to numerically evaluate $H(K, j; a, b, c)$. We will use this algorithm as the building block for a method to compute zeta. Given this roadmap, we prove

Theorem 2.2. *There exist absolute constants κ_3 , κ_4 , and κ_5 such that for any $\mu \in [0, 1]$, any $\epsilon \in (0, e^{-1})$, any positive integer K , any non-negative integer*

j , any $a, b \in \mathbb{R}/\mathbb{Z}$, any $c \in [0, K^{-(3-\mu)}]$, and with an underlying computational model that performs arithmetic operations using $O(\nu^2)$ -bit arithmetic, where $\nu = \nu(K, j, \epsilon) := (j+1) \log(K/\epsilon)$, the function $H(K, j; a, b, c)$ can be computed with absolute error bounded by $O(\nu^{\kappa_3} \epsilon)$ using $O(\nu^{\kappa_4})$ arithmetic operations provided a precomputation costing $O(\nu^{\kappa_5} K^{4\mu})$ operations is performed. The big- O constants are absolute.

In Theorems 2.1 and 2.2 a bit complexity bound follows routinely from the arithmetic operations bound because all the numbers that occur in the algorithms have $O(\nu^2)$ bits. We did not try to obtain numerical values for the constants κ_3 , κ_4 , and κ_5 in Theorem 2.2. With some optimization, they probably could be taken to be around 4. Also, in a practical version of the algorithm the arithmetic can be performed using substantially fewer than $O(\nu^2)$ bits and we would likely be able to replace $\nu(K, j, \epsilon)$ by $j + \log(K/\epsilon)$.

Theorem 2.2 yields our $t^{4/13+o(1)}$ zeta method as follows. First, choose $\beta = 5/26$. Then by the argument at the beginning of this section, we only need to deal with sums of the form

$$\begin{aligned} H(K, j; a, b, c), \quad a, b \in \mathbb{R}/\mathbb{Z}, \\ 0 \leq c \leq t/N^3, \quad 0 \leq j = O(\log t) \end{aligned}$$

Assume $K > 1$. Since $K \leq N t^{5/26-1/2}$, then $c \leq t^{1/13}/K^3$. Let μ be the solution of the equation $t^{1/13} = K^\mu$. If $\mu \geq 1$, then $K \leq t^{1/13}$. So, we can use FFT to precompute the functions

$$H(K, j; \tilde{a}, \tilde{b}, \tilde{c}), \quad 0 \leq j = O(\log t)$$

at all points $(\tilde{a}, \tilde{b}, \tilde{c})$ of the lattice

$$\{(\delta/K, \tau/K^2, \gamma/K^3) \mid 0 \leq \delta < K, 0 \leq \tau < K^2, 0 \leq \gamma \leq t^{1/13}\}$$

The cost of the FFT precomputation is $K^3 t^{1/13+o(1)} \leq t^{4/13+o(1)}$. Once the precomputation is carried out, Taylor expansions can be used to evaluate the function $H(K, j; a, b, c)$ in $t^{o(1)}$ operations. If $\mu < 1$, then apply the algorithm for cubic sums. According to this algorithm, we merely need to perform an FFT precomputation costing $O(K^{4\mu+o(1)}) = O(t^{4/13+o(1)})$ operations. Lastly, recall the error function $R(t)$ in the Riemann-Siegel formula (1). Then put together, we have obtained

Theorem 2.3. *There exists an absolute constant κ_6 such that for any $t > 10^6$ and with an underlying computational model that performs arithmetic operations using $O((\log t)^2)$ -bit arithmetic, the function $\zeta(1/2 + it)$ can be computed with error $R(t)$ using $O((\log t)^{\kappa_6} t^{4/13})$ arithmetic operations. The big- O constants are absolute.*

Our $t^{4/13+o(1)}$ -method appears to be compatible with the amortized complexity techniques of Odlyzko-Schönhage [OS]. This means it is possible to modify it so as to permit the computation of $O(T^{4/13})$ values of $\zeta(1/2 + iT + t)$

for $t \in [0, T^{4/13}]$ in $T^{4/13+o(1)}$ time. It also appears if certain quite substantial modifications to our cubic sums algorithm are carried out, then the $t^{4/13+o(1)}$ method is capable of improvement to $t^{3/10+o(1)}$. These issues will be explored in a separate paper [H2].

Finally, we remark that the essential ingredients in the aforementioned methods are quite independent from the zeta function itself. They employ specific properties of $\zeta(1/2 + it)$ only in so far as they rely on the Riemann-Siegel formula as a starting point. In the case of our $t^{1/3+o(1)}$ method, the use of the Riemann-Siegel formula is not critical. In fact, one can still achieve $t^{1/3+o(1)}$ complexity starting with the Euler-Maclaurin formula.

3 An informal description of the algorithm to compute cubic exponential sums

The general idea behind Theorem 2.2 is to extend the basic approach developed in [H1] to compute quadratic (theta) sums. That approach depends on two main ingredients; the periodicity of the complex exponential and the self-similarity of the Gaussian. By the self similarity of the Gaussian we mean the property

$$\int_{-\infty}^{\infty} e^{\alpha t - t^2} dt = \sqrt{\pi} e^{\alpha^2/4}, \quad \alpha \in \mathbb{C}$$

In the case of cubic sums, we still have the periodicity but not the self-similarity. However, if the cubic coefficient c is relatively small, then there is approximate self-similarity. This permits repeated applications of Poisson summation, much like what occurs in the quadratic case. The length of the cubic sum is cut by at least one half after each application of Poisson summation, but also the magnitude of the cubic coefficient grows. So, after a certain number of iterations, the approximate self-similarity weakens and it is no longer possible to simulate the algorithm for quadratic sums. At this point, we stop iterating and we use saddle-point methods to reduce the problem to evaluating sums of the form

$$\frac{1}{M^t} \sum_{k=0}^M k^l \exp(2\pi i \beta_1 k + 2\pi i \beta_2 k^2 + \dots + 2\pi i \beta_S k^S) \quad (17)$$

where $3 \leq S = O(\log K)$. Finally, the Fast Fourier transform is employed to precompute a large number of sums of the form (17).

We elaborate. The initial step in our algorithm is to convert the cubic sum to an integral. This is followed by two “while loops,” a transformation, another “while loop,” and finally an FFT precomputation. We overview this chain of steps in more detail in the next few paragraphs. Before doing so however, we digress to comment that this chain of steps is by no means the only path to implementing the essential features of the algorithm for cubic sums. In particular, although our initial reformulation of the cubic sums as an integral dictates many technical structures in the remainder of the paper, it is

completely avoidable. The reason we choose this implementation is because it allows us to avoid reconstructing the algorithm for quadratic sums from scratch.

On to a description of the algorithm for cubic sums. Let $\epsilon \in (0, e^{-1})$. Recall our assumption in Theorem 2.2 that $c \in [0, K^{-(3-\mu)}]$. Rewrite $H(K, j; a, b, c)$ as the integral

$$\frac{1}{N^j} \int_{-1/2}^{1/2} F(K; a - \alpha x, b) Q(K, j; x, c) dx$$

where

$$F(K; a, b) := F(K, 0; a, b), \quad Q(K, j; x, c) := \frac{1}{K^j} \sum_{k=0}^K k^j \exp(2\pi i a k + 2\pi i c k^3)$$

and initially $\alpha = 1$. In the first loop, the quadratic and cubic sums F and Q are partially “decoupled”. The extent of decoupling is measured by the parameter α , which remains positive throughout. The smaller the value of α at the end of first loop, the more independent is the quadratic sum $F(a - \alpha x, b)$ from the cubic sum $Q(K, j; x, c)$. With each iteration in the first loop the size of c grows while the length K of the cubic sum decreases. The loop is terminated when we near the region where c is too large to efficiently apply Euler-Maclaurin summation to $Q(K, j; x, c)$. Once the first loop is terminated, Q is converted to an integral. This yields the expression

$$\frac{1}{N^j} \int_0^N \exp(2\pi i c y^3) \int_{-1/4}^{1/4} \exp(-2\pi i x y) F(K; a - \alpha x, b) dx$$

where $\alpha = O(K^{-(1-\mu)})$, $N = O(K^\mu)$, and $c = O(1/N^2)$. In the second loop, the algorithm for quadratic sums is applied to $F(a - \alpha x, b)$. It is possible to do so despite the presence of an integral sign with respect to x precisely because α is relatively small. With each iteration of the algorithm for quadratic sums, the value of α increases, while the length K of the quadratic sum decreases. The loop is terminated when α exceeds $\nu(K, j, \epsilon)^{-6}$, at which point the cost of further applications of the quadratic algorithm starts becoming significant. At the end of the second loop, we are left with an expression of the form

$$\frac{1}{N^j} \int_0^N \exp(2\pi i c y^3) \int_{-1/4}^{1/4} \exp(-2\pi i y x - 2\pi i \alpha_1 x - 2\pi i \alpha_2 x^2) F(M; a - \alpha x, b) dx$$

where $\alpha > \nu(K, j, \epsilon)^{-6}$, $M \leq N/\alpha$, and α_1, α_2 are certain parameters that are related to α . Some elementary saddle-point calculations are used to reduce the last expression to sums of the form

$$\frac{1}{M^l} \sum_{k=0}^M k^l \exp(2\pi i \beta_1 k + 2\pi i \beta_2 k^2 + \dots + 2\pi i \beta_S k^S) \quad (18)$$

where $3 \leq S \leq 3 + \log N / \log M$ and $0 \leq l = O(\nu(K, j, \epsilon))$ are integers. In the third loop, we perform a dyadic approximation of M . This reduces the problem

to computing sums of the form

$$\frac{1}{2^{nl}} \sum_{k=0}^{2^n} k^l \exp(2\pi i \beta_1 k + 2\pi i \beta_2 k^2 + \dots + 2\pi i \beta_S k^S) \quad (19)$$

where $2^n = O(M)$ and $0 \leq l = O(\nu(K, j, \epsilon))$ is an integer. There are some relations among the coefficients β_{s+3} , $s \geq 1$ which we exploit in the course of performing an FFT precomputation of the sums (19). The overall cost of FFT is $N^{4+o(1)} = K^{4\mu+o(1)}$. After the precomputation, the sums (19) can be evaluated quickly via Taylor expansions. This concludes our sketch of the algorithm for cubic sums.

From our discussion in Section 2, it is not hard to deduce that the value of μ relevant to computing a sum of the form

$$\sum_{n=N}^{2N} \frac{\exp(\log n)}{\sqrt{n}}$$

is given by

$$\mu = \frac{(1/13) \log t}{\log N - (4/13) \log t}$$

In particular, if $N = t^{1/2}$, then $\mu = 2/5$. For this reason, as well as for the sake of definiteness, we prove Theorem 2.2 only in the case $\mu = 2/5$. It will be clear however that the proof holds virtually unmodified for any $0 \leq \mu \leq 1$ (the absolute constants implied by our proof are independent of μ).

4 The algorithm for $H(K, j; a, b, c)$

Let $\epsilon \in (0, e^{-1})$, $j \geq 0$, and $K > 0$. Define $\nu(K, j, \epsilon) := (j+1) \log(K/\epsilon)$. We say K is *large enough* if $K > \Lambda(K, j, \epsilon)$ where $\Lambda(K, j, \epsilon) := 50^4 \nu(K, j, \epsilon)^6$. We also define $F(N; a, b) := F(N, 0; a, b)$ and

$$Q(N, j; a, c) := \frac{1}{K^j} \sum_{k=0}^N k^j \exp(2\pi i a k + 2\pi i c k^3)$$

In this section, the quantities K , j , and ϵ are treated as global constants, by which we mean different occurrences of them will denote the same values. For this reason, we drop the dependence of Λ and ν on K , j , and ϵ throughout Section 4. We use the same computational model as the one described at the beginning of Section 2 except now arithmetic is performed using $O(\nu(K, j, \epsilon)^2)$ -bits

Finally, in order to spare the reader some mundane and other straightforward details, we sometimes use informal phrases such as “It is possible to *reduce/simplify* the problem of computing the function $X_{K,j}(\cdot)$ to that of computing the function $Y_{K,j}(\cdot)$,” or “In order to compute the function $X_{K,j}(\cdot)$, it

is enough/it suffices/we just need to compute the function $Y_{K,j}(\cdot)$.” This means there exist absolute constants $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ such that, for any $\epsilon \in (0, e^{-1})$, once the function $Y_{K,j}(\cdot)$ is computed for the relevant values of its arguments with error bounded by $O(\epsilon)$, then the function $X_{K,j}(\cdot)$ can be computed for the relevant values of its arguments with error bounded by $O(\nu(K, j, \epsilon)^{\tilde{\kappa}_1} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\tilde{\kappa}_2})$ arithmetic operations. The meaning of the phrase “relevant values of the arguments” will be clear from the context. We frequently say “the function $X_{K,j}(\cdot)$ can be *computed/evaluated/handled efficiently/quickly/easily*.” As one might expect by now, this means there exist absolute constants $\tilde{\kappa}_3$ and $\tilde{\kappa}_4$ such that, for any $\epsilon \in (0, e^{-1})$, the function $X_{K,j}(\cdot)$ can be computed for the relevant values of its arguments with error bounded by $O(\nu(K, j, \epsilon)^{\tilde{\kappa}_3} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\tilde{\kappa}_4})$ arithmetic operations.

4.1 The first loop: decouple the cubic and quadratic sums

Assume that K is large enough, otherwise the problem trivializes. Let $c \in [0, K^{-13/5}]$. Define

$$S_{a,b,j}^{K,p}(N, c, \alpha) := \frac{1}{N^j} \int_{x_p}^{y_p} F(K; a - \alpha x, b) Q(N, j; x, c) dx$$

where

$$\begin{aligned} (x_1, y_1) &:= (-1/2, -1/4), & (x_2, y_2) &:= (-1/4, 1/4) \\ (x_3, y_3) &:= (1/4, 1/2), & (x_4, y_4) &:= (1/4, 3/4) \end{aligned} \quad (20)$$

Then

$$H(K, j; a, b, c) = S_{a,b,j}^{K,1}(K, c, 1) + S_{a,b,j}^{K,2}(K, c, 1) + S_{a,b,j}^{K,3}(K, c, 1)$$

By lemmas 5.1 through 5.4 in Section 5, the functions $S_{a,b,j}^{K,1}(K, c, 1)$ and $S_{a,b,j}^{K,3}(K, c, 1)$ can be computed quickly. So, it is enough to show how to evaluate $S_{a,b,j}^{K,2}(K, c, 1)$. We may assume that K is a power of 2. Define $N_m := K/2^m$, $c_m := 2^{3m}c$, and $\alpha_m := 2^{-m}$. Since

$$Q(N_m, j; x, c_m) = 2Q(N_{m+1}, j, 2x, c_{m+1}) - Q(N_m, j; x + 1/2, c_m)$$

then using the change of variable $x \rightarrow 2x$, we obtain

$$S_{a,b,j}^{K,2}(N_m, c_m, \alpha_m) = S_{a,b,j}^{K,2}(N_{m+1}, c_{m+1}, \alpha_{m+1}) + R_{a,b,j}^K(N_m, c_m, \alpha_m) \quad (21)$$

where

$$\begin{aligned} R_{a,b,j}^K(N_m, c_m, \alpha_m) &= S_{a,b,j}^{K,1}(N_{m+1}, c_{m+1}, \alpha_{m+1}) + S_{a,b,j}^{K,3}(N_{m+1}, c_{m+1}, \alpha_{m+1}) \\ &\quad - S_{a+\alpha_{m+1}, b, j}^{K,4}(N_m, c_m, \alpha_m) \end{aligned}$$

By lemmas 5.1 through 5.4, the function $R_{a,b,j}^{K,2}(N_m, c_m, \alpha_m)$ can be computed efficiently. Thus, we may iterate the relation (21). After at most $O(\log K)$

iterations, either $c_m N_m^2 > 1/\Lambda$ or N_m is not large enough, at which points we stop iterating. The latter case is a boundary point of the algorithm; the problem simplifies to computing the sums

$$\sum_{k=1}^K \exp(2\pi i a k + 2\pi i b k^2) \int_{-1/2}^{1/2} \exp(2\pi i x n - 2\pi i \alpha_m k x) dx \quad (22)$$

where $0 \leq n \leq N_m = O(\Lambda)$ is an integer. By the change of variable $x \rightarrow x + 1/2$, the sums (22) can be reduced to the form

$$\int_0^1 e^{2\pi i n x} F(K; \tilde{a} + \beta x, b) dx \quad (23)$$

where $\beta \in [-1, 1]$. By Lemma 5.10, functions of the form (23) can be evaluated efficiently. So put together, we may assume N_m is large enough and that the first loop is terminated due to a violation of the condition $0 \leq c_m N_m^2 \leq 1/\Lambda$.

If c_m no longer satisfies the condition $0 \leq c_m N_m^2 \leq 1/\Lambda$, then $c_m N_m^2 > 1/\Lambda$. Observe

$$1/\Lambda \leq c_m N_m^2 \Rightarrow 1/\Lambda \leq (2^{3m} c)(K/2^m)^2 \Rightarrow \alpha_m \leq \Lambda c K^2$$

So recalling $0 \leq c \leq K^{-13/5}$, it follows $\alpha_m \leq \Lambda K^{-3/5}$. This in turn implies that $N_m = \alpha_m K \leq \Lambda K^{2/5}$.

To summarize, let $N := N_m$, $\alpha := \alpha_m$, and re-define $c := c_m$. Assume $K^{3/5} \geq \Lambda^2$, otherwise, the sum $H(K, j; a, b, c)$ has length $O(\Lambda^{10/3})$. So, it can be computed directly. Our problem has then been reduced to evaluating

$$S_{a,b,j}^{K,2}(N, c, \alpha) = \frac{1}{N^j} \int_{-1/4}^{1/4} F(K; a - \alpha x, b) Q(N, j; x, c) dx$$

with conditions

$$\begin{aligned} 1/\Lambda \leq c N^2 \leq 2/\Lambda, \quad \Lambda \leq N \leq \Lambda K^{2/5} \\ \Lambda/K < \alpha \leq \Lambda K^{-3/5} \leq 1/\Lambda \end{aligned} \quad (24)$$

4.2 The second loop: apply the algorithm for quadratic sums to $F(K; a + \alpha x, b)$

By lemmas 5.1 through 5.3 in Section 5 we have

$$\begin{aligned} S_{a,b,j}^{K,2}(N, c, \alpha) &= \frac{1}{N^j} \int_{-1/4}^{1/4} \int_0^N y^j \exp(2\pi i c y^3 - 2\pi i x y) F(K; a + \alpha x, b) dy dx \\ &\quad + E_{a,b,j}^K(N, c, \alpha) \end{aligned}$$

and $E_{a,b,j}^K(N, c, \alpha)$ can be computed efficiently. So, we only need to show how to evaluate the function

$$\frac{1}{N^j} \int_0^N \int_{-1/4}^{1/4} y^j \exp(2\pi i c y^3 - 2\pi i x y) F(K; a + \alpha x, b) dx dy \quad (25)$$

where N , c , and α satisfy assumptions (24) and $a, b \in \mathbb{R}/\mathbb{Z}$.

The purpose of this section is to show how to apply the van der Corput iteration to the function $F(K; a + \alpha x, b)$ in (25). In order to do so, we need to introduce some notation. To better understand the motivation for the notation, one will find it helpful to review lemmas 6.6 and 6.7 in [H1] first. In particular, recall the relation

$$\begin{aligned} F(\tilde{K}; \tilde{a} + \tilde{\alpha}x, \tilde{b}) &= \frac{e^{\pi i/4 - \pi i \tilde{a}^2/(2\tilde{b})}}{\sqrt{2\tilde{b}}} e^{-2\pi i \frac{\tilde{\alpha}\tilde{a}}{2\tilde{b}}x - 2\pi i \frac{\tilde{\alpha}^2}{4\tilde{b}}x^2} F\left(\lfloor 2\tilde{b}\tilde{K} \rfloor; \frac{\tilde{a}}{2\tilde{b}} + \frac{\tilde{\alpha}x}{2\tilde{b}}, -\frac{1}{4\tilde{b}}\right) \\ &\quad + R(\tilde{K}, \tilde{a} + \tilde{\alpha}x, \tilde{b}) + O(e^{-\tilde{K}}), \end{aligned} \quad (26)$$

which is valid for any integer $\tilde{K} > 1000$, any $x \in [-1/4, 1/4]$, and any tuple $(\tilde{\alpha}, \tilde{a}, \tilde{b}) \in [-1, 1] \times [0, 2] \times [0, 1/4]$ satisfying

$$[\tilde{a} + \tilde{\alpha}x] < \lfloor 2\tilde{b}\tilde{K} + \tilde{a} + \tilde{\alpha}x \rfloor \quad (\tilde{a} + \tilde{\alpha}x, \tilde{b}) \in (0, 2) \times [0, 1/4]$$

for all $x \in [-1/4, 1/4]$. The structure of the function $R(\tilde{K}, \tilde{a} + \tilde{\alpha}x, \tilde{b})$ is fully described by lemmas 6.6 and 6.7 in [H1]; see (32). We wish to learn the outcome of repeated applications of relation (26). With this in mind, we proceed to define our notation.

For $z \in \mathbb{C}$ and $\theta \in \{-1, 1\}$, define

$$\Phi_\theta(z) = \begin{cases} \bar{z} & \text{if } \theta = -1 \\ z & \text{if } \theta = 1 \end{cases}$$

By the periodicity of the complex exponential, for reals \tilde{a} , \tilde{b} , $\tilde{\alpha}$, and integers \tilde{K} , we have

$$F(\tilde{K}; \tilde{a} + \tilde{\alpha}x \pm 1/2, \tilde{b} \pm 1/2) = F(\tilde{K}; \tilde{a} + \tilde{\alpha}x, \tilde{b})$$

Thus, for $\tilde{K} > 0$, there exists exactly one or two tuples $(\tilde{a}_0, \tilde{b}_0, \theta) \in \mathbb{R}^2 \times \{-1, 1\}$, which depend only on (\tilde{a}, \tilde{b}) , such that

$$\begin{aligned} F(\tilde{K}; \tilde{a} + \tilde{\alpha}x, \tilde{b}) &= \Phi_\theta(F(\tilde{K}; \tilde{a}_0 + \theta\tilde{\alpha}x, \tilde{b}_0)), \\ (\tilde{a}_0 + \theta\tilde{\alpha}x, \tilde{b}_0) &\in (0, 2) \times [0, 1/4], \quad \text{for all } x \in [-1/4, 1/4] \end{aligned} \quad (27)$$

Given a real pair (\tilde{a}, \tilde{b}) , define $(\tilde{a}_0, \tilde{b}_0, \theta)$ be the tuple satisfying (27) with the least possible \tilde{a}_0 . Since $(\tilde{a}_0, \tilde{b}_0, \theta)$ depends only on (\tilde{a}, \tilde{b}) , we may write it in the form

$$\Psi(\tilde{a}, \tilde{b}) := (\tilde{a}_0, \tilde{b}_0, \theta)$$

Also, define the function

$$\Upsilon(\tilde{a}, \tilde{b}, \theta, \tilde{\alpha}) := \left(\frac{\theta\tilde{a}}{2\tilde{b}}, \frac{-\theta}{4\tilde{b}}, \frac{\tilde{\alpha}}{2\tilde{b}} \right), \quad \tilde{b} \neq 0$$

Let $(\tilde{a}_0, \tilde{b}_0) := (a, b)$, $\alpha_{-1} := \alpha$, and for integers $l \geq 0$ sequentially define

$$(a_l, b_l, \theta_l) := \Psi(\tilde{a}_l, \tilde{b}_l), \quad (\tilde{a}_{l+1}, \tilde{b}_{l+1}, \alpha_l) := \Upsilon(a_l, b_l, \theta_l, \alpha_{l-1})$$

In addition, let $\alpha_{1,-1} := 0$, $\alpha_{2,-1} := 0$, $K^{(0)} := K$, and for integers $l \geq 0$ sequentially define

$$\alpha_{1,l} = \alpha_{1,l-1} + a_l \alpha_l, \quad \alpha_{2,l} = \alpha_{2,l-1} + \theta_l b_l \alpha_l^2, \quad K^{(l)} := \lfloor 2b_l K^{(l-1)} \rfloor$$

Lastly, let $m \in [-1, \infty)$ be the least integer for which at least one of the following conditions is violated for some $x \in [-1/4, 1/4]$

$$\begin{aligned} \alpha_m &\leq 1/\Lambda, & K^{(m+1)} &> \Lambda, \\ \lceil a_{m+1} + \theta_{m+1} \alpha_m x \rceil &< \lfloor a_{m+1} + \theta_{m+1} \alpha_m x + 2b_{m+1} K^{(m+1)} \rfloor \end{aligned} \quad (28)$$

Before we continue, we want to make some remarks. Since $2b_l \leq 1/2$, then $K^{(l+1)} \leq K^{(l)}/2$. Thus, by construction $m = O(\log K)$. Note that the third condition in (28) implies $2b_l \geq 1/K^{(l)}$ for all integers $0 \leq l \leq m$. Also, for $0 \leq l \leq m$, we have

$$\begin{aligned} \alpha_{1,l} &< 2\alpha_l \sum_{r=0}^l 2^{-r} \leq 4\alpha_l, & |\alpha_{2,l}| &\leq \frac{\alpha_l}{2} \sum_{r=0}^l \alpha_{l-1} \leq \alpha_l, & \alpha_{l-1} &\leq 1/\Lambda \\ \alpha_{l-1} K^{(l)} &\leq N = \alpha K \leq \alpha_{l-1} K, & \alpha_{1,l}/\alpha_l &= O(1), & \alpha_{2,l}/\alpha_l^2 &= O(1) \end{aligned} \quad (29)$$

This set of observations will be useful later in this section.

With the notation introduced so far, the second loop of our algorithm can be described easily. The loop consists of exactly $m+1$ applications of Lemma 6.7 in [H1] to the function $F(K; a + \alpha x, b)$; that is, exactly $m+1$ applications of relation (26). The final output of the process has the form

$$\begin{aligned} F(K; a + \alpha x, b) &= C_m e^{-2\pi i \alpha_{1,m} x - 2\pi i \alpha_{2,m} x^2} F(K^{(m+1)}; \tilde{a}_{m+1} + \alpha_m x, \tilde{b}_{m+1}) \\ &\quad + \tilde{R}_m(K; a, \alpha, x, b) + O(e^{-K^{(m)}}) \end{aligned} \quad (30)$$

where

$$\tilde{R}_m(K; a, \alpha, x, b) := \sum_{l=0}^m C_{l-1} e^{-2\pi i \alpha_{1,l-1} x - 2\pi i \alpha_{2,l-1} x^2} \Phi_{\theta_l}(R(K^{(l)}, a_l + \theta_l \alpha_{l-1} x, b_l))$$

Here, $C_{-1} = 0$, and the constants C_l , $0 \leq l \leq m$, are independent of x , computable in $O(1)$ operations, and are bounded by $O(\sqrt{K})$.

In order for the relation (30) to possibly be useful, we need to be able to efficiently compute the functions

$$\begin{aligned} \frac{1}{N^j} \int_{-1/4}^{1/4} \int_0^N y^j e^{2\pi i c y^3 - 2\pi i x y} e^{-2\pi i \alpha_{1,l-1} x - 2\pi i \alpha_{2,l-1} x^2} \times \\ \Phi_{\theta_l}(R(K^{(l)}, a_l + \theta_l \alpha_{l-1} x, b_l)) dy dx \end{aligned}$$

for $0 \leq l \leq m$. Fix l , so we may drop dependencies on it. Let (w, z) be a subinterval of $[-1/4, 1/4]$ over which

$$\lfloor a_l + \theta_l \alpha_{l-1} x + 2b_l K^{(l)} \rfloor \text{ and } \lceil a_l + \theta_l \alpha_{l-1} x \rceil \text{ are constant for all } x \in (w, z)$$

Since $\alpha_{l-1} = O(1)$, the interval $[-1/4, 1/4]$ can be written as the closure of the union of finitely many such subintervals. So, we can restrict our attention to the interval (w, z) . By lemmas 6.7 and 6.8 in [H1], there exist four constants $\{\lambda_i\}_{1 \leq i \leq 4}$, and four integers $\tilde{\theta}_i \in \{-1, 1\}$ satisfying

$$0 \leq \lambda_i + \tilde{\theta}_i \alpha_{l-1} x \leq 1, \quad \text{for all } x \in (w, z) \text{ and } 1 \leq i \leq 4, \quad (31)$$

such that $R(K^{(l)}, a_l + \theta_l \alpha_{l-1} x, b_l)$ is equal to a linear combination of the functions

$$\begin{aligned} & x^r, \quad x^r e^{2\pi i \theta_l \alpha_{l-1} K^{(l)} x}, \quad e^{2\pi i \theta_l \alpha_l B x - 2\pi i \frac{\alpha_{l-1}^2}{4b_l} x^2}, \\ & (\lambda + \theta \alpha_{l-1} x)^r e^{2\pi i L(\lambda + \theta \alpha_{l-1} x) - 2\pi R(\lambda + \theta \alpha_{l-1} x)}, \\ & e^{2\pi i Q(\lambda + \theta \alpha_{l-1} x) - 2\pi(1-i)r \frac{\lambda + \theta \alpha_{l-1} x}{\sqrt{2b_l}}} \int_0^1 t^d e^{-2\pi(1-i) \frac{\lambda + \theta \alpha_{l-1} x}{\sqrt{2b_l}} t - 2\pi r t} dt \end{aligned} \quad (32)$$

where $(\lambda, \theta) \in \{(\lambda_i, \tilde{\theta}_i)\}_{1 \leq i \leq 4}$, plus an error bounded by $O(\Lambda K^{-2}\epsilon)$. Here, the integers B, L, R, Q, d , and r satisfy

$$\begin{aligned} Q &\in \{0, K^{(l)}\}, & B &\in \{-1, 0, K^{(l+1)}, K^{(l+1)} + 1\}, \\ 0 &\leq p = O(\nu), & K^{(l)} &\leq L \leq K^{(l)} + p \\ 0 &\leq q = O(\nu), & K^{(l)} &\leq R \leq K^{(l)} + q \\ 0 &\leq d = O(\nu), & 0 &\leq r = O(\nu) \end{aligned}$$

The length of the linear combination is $O(\Lambda)$. The coefficients in the linear combination can all be computed in $O(\Lambda)$ operations, are bounded by $O(K)$, and of course are independent of x . Also, big- O constants are absolute.

If $f(x)$ assumes any of the first three forms in (32), then by Lemma 5.5, the integral

$$\frac{1}{N^j} \int_0^N \int_w^z y^j e^{2\pi i c y^3 - 2\pi i x y} e^{-2\pi i \alpha_{1,l-1} x - 2\pi i \alpha_{2,l-1} x^2} \Phi_{\theta_l}(f(x)) dx dy \quad (33)$$

can be computed efficiently. If $f(x)$ assumes the fourth expression in (32), then make the change of variable $\lambda + \theta \alpha_{l-1} x \rightarrow x$ followed by $y \rightarrow y/\alpha_{l-1}$. This yields an integral of the form

$$\frac{1}{N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta y} \int_{w_1}^{z_1} x^r e^{2\pi i (\tau \pm y)x - 2\pi i \beta x^2 - 2\pi R x} dx dy \quad (34)$$

where $K^{(l)} < N_1 \leq K$, $c_1 N_1^2 \leq 1/\Lambda$, $(w, z) \subset [0, 1]$, and, by set of observations (29), the real numbers β, η, τ are bounded by $O(1), O(1), O(N_1)$ respectively. Once again, Lemma 5.5 shows that the integral (34) can be evaluated efficiently. In fact, the Lemma proves a much more general statement.

Lastly, suppose $f(x)$ assumes the fifth expression in (32). For the sake of clarity, let us assume $r = 0$ and $d = 0$ in that expression. The treatment of the case when r or d is positive is similar. Once again, we make the change of

variable $\lambda + \theta\alpha_{l-1}x \rightarrow x$ followed by $y \rightarrow y/\alpha_{l-1}$. This yields an integral of the form

$$\frac{1}{N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta y} \int_{w_1}^{z_1} e^{2\pi i (\tau_1 \pm y)x - 2\pi i \beta x^2} \int_0^1 e^{-\frac{2\pi(1 \pm i)x t}{\sqrt{2b_l}}} dt dx dy \quad (35)$$

where τ_1 is a real number bounded by $O(N_1)$. The treatment of the integral (35) is completely elementary, albeit quite tedious. This is the reason we relegate the evaluation of many of the integrals that occur in this paper to Section 5. Despite this, in the next few paragraphs, we show how one can begin to tackle the exponential integral (35). This might give some idea of the type of manipulations we use to simplify such integrals. Note however, the actual evaluation of these integrals employs some basic saddle-point techniques. Although we do not discuss this in the present section, it is discussed in detail in Sections 4.3 and 5.

So, first observe we may assume $w_1 \geq \Lambda\sqrt{2b_l}$ in (35), otherwise, for all $x \in [w_1, \Lambda\sqrt{2b_l}]$, the cross term $e^{-2\pi(1 \pm i)x t / \sqrt{2b_l}}$ can be routinely eliminated via appropriate changes of variable and Taylor expansions, which yields integrals of the type handled by Lemma 5.5. We integrate with respect to t in (35) to obtain integrals of the form

$$\frac{\sqrt{2b_l}}{N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta y} \int_{w_1}^{z_1} \frac{1}{x} e^{2\pi i (\tau_2 \pm y)x - 2\pi i \beta x^2 - 2\pi \gamma x} dx dy \quad (36)$$

where $\tau_2 = O(N_1)$ is a real number, and $\gamma \in \{0, 1/\sqrt{2b_l}\}$. Since the case $\gamma = 1/\sqrt{2b_l}$ is negligible, we can take $\gamma = 0$ in (36). Apply the change of variable $x \rightarrow x/w_1$, and note $\sqrt{2b_l}/w_1 = O(1)$. Then, we arrive at

$$\frac{1}{N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta y} \int_1^{z_1/w_1} \frac{1}{x} e^{2\pi i w_1 (\tau_2 \pm y)x - 2\pi i \beta w_1^2 x^2} dx dy \quad (37)$$

By hypothesis, $z_1/w_1 \leq \sqrt{K^{(l)}}$. So, the integral (37) can be subdivided into $O(\log z_1/w_1) = O(\log K)$ consecutive subintegrals of the form

$$\frac{1}{N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta y} \int_A^{A+\Delta} \frac{1}{x} e^{2\pi i w_1 (\tau_2 \pm y)x - 2\pi i \beta w_1^2 x^2} dx dy \quad (38)$$

where $1 \leq A \leq z_1/w_1$ and $0 < \Delta \leq A/2$. Next, we make the change of variable $x \rightarrow x - A$ in (38). This replaces x^{-1} with $(x + A)^{-1}$. We factor out A from $(x + A)^{-1}$, and then apply Taylor expansions to the resulting term $(1 + x/A)^{-1}$. The said series of manipulations reduces (38) to integrals of the form

$$\frac{1}{A^{s+1} N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta y} \int_0^\Delta x^s e^{2\pi i w_1 (\tau_3 \pm y)x - 2\pi i \beta w_1^2 x^2} dx dy \quad (39)$$

where $0 \leq s = O(\nu)$ is an integer, and $\tau_3 = O(N_1)$, $\eta_1 = O(\sqrt{N_1})$ are real numbers. To conclude, make the change of variable $w_1 x \rightarrow x$. This leads to an

integral of the form

$$\frac{1}{(w_1 A)^{s+1} N_1^j} \int_0^{N_1} y^j e^{2\pi i c_1 y^3 + 2\pi i \eta_1 y} \int_0^{w_1 \Delta} x^s e^{2\pi i (\tau_3 \pm y)x - 2\pi i \beta x^2} dx dy \quad (40)$$

The integral (40) can essentially be handled via Lemma 5.5.

Finally, Lemma 5.5 shows how to deal with a violation of the second condition in (28). Furthermore, a violation of the third condition in (28) can be handled by appealing to Lemma 5.9. In particular, we may assume when the quadratic algorithm terminates that $K^{(m+1)} > \Lambda$ and $\alpha_m > 1/\Lambda$.

To summarize, let $M := K^{(m+1)}$, $\alpha := \alpha_m$, $\alpha_1 := \alpha_{1,m}$, and $\alpha_2 := \alpha_{2,m}$. We can assume $m \geq 0$, for else, the third condition in (28) is violated for $m = -1$, in which case Lemma 5.9 takes care of the situation. A simple estimate then yields $|\alpha_{2,m}| = |\alpha_2| > 1/K^2$. So put together, our problem has been reduced to evaluating

$$\frac{1}{N^j} \int_0^N \int_{-1/4}^{1/4} y^j \exp(2\pi i c y^3 - 2\pi i x y) \exp(2\pi i \alpha_2 x^2 - 2\pi i \alpha_1 x) \times \\ F(M; a + \alpha x, b) dx dy \quad (41)$$

where

$$N \leq \Lambda K^{2/5}, \quad \Lambda \leq M \leq N/\alpha, \quad 1/\Lambda \leq \alpha \leq N/M \\ 0 \leq \alpha_1 \leq 4\alpha, \quad 1/\Lambda \leq cN^2 \leq 2/\Lambda, \quad 1/K^2 \leq |\alpha_2| \leq \alpha \quad (42)$$

4.3 Some saddle-point calculations

We want to compute (41) with conditions (42). Let us explicitly write (41) as

$$\frac{1}{N^j} \int_0^N \int_{-1/4}^{1/4} y^j \exp(2\pi i c y^3 - 2\pi i x y) \exp(2\pi i \alpha_2 x^2 - 2\pi i \alpha_1 x) \times \\ \sum_{k=0}^M \exp(2\pi i (a + \alpha x)k + 2\pi i b k^2) dx dy \quad (43)$$

Define $T := \lfloor \Lambda^2 \rfloor$. Then, split the sum over k in (43) into three possibly empty subsums: one that consists of the terms $T < k < M - T$, another that consists of $0 \leq k \leq T$, and a third subsum consisting of $M - T \leq k \leq M$. By Lemma 5.5, the last two subsums can be computed quickly. We can assume the first subsum is not empty, otherwise, the algorithm terminates. By Lemma 5.6, for each term in the first subsum, the domain of integration with respect to x can be extended to $(-\infty, \infty)$. Each of these terms thus becomes

$$\frac{1}{N^j} \int_0^N y^j \exp(2\pi i c y^3) \int_{-\infty}^{\infty} \exp(2\pi i \alpha k x - 2\pi i x y - 2\pi i \alpha_1 x + 2\pi i \alpha_2 x^2) dx dy \\ = \frac{\exp(\operatorname{sgn}(\alpha_2) \frac{i\pi}{4})}{\sqrt{2|\alpha_2|} N^j} \int_0^N y^j \exp(2\pi i f_k(y)) dy \quad (44)$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and

$$f_k(y) = f_k(y, c, \alpha, \alpha_1, \alpha_2) := cy^3 - \frac{y^2}{4\alpha_2} + \frac{\alpha k - \alpha_1}{2\alpha_2}y - \frac{(\alpha k - \alpha_1)^2}{4\alpha_2}$$

Here, the easily-provable formula

$$\int_{-\infty}^{\infty} e^{2\pi i x t + 2\pi i y t^2} dt = \frac{e^{\operatorname{sgn}(y)\pi i/4}}{\sqrt{2|y|}} e^{-2\pi i x^2/(4y)}, \quad x, y \in \mathbb{R}, y \neq 0$$

was used. We want to extract the saddle point contribution from (44). To this end, define

$$y_k = y_k(c, \alpha, \alpha_1, \alpha_2) := \frac{1}{12c\alpha_2} \left(1 - \sqrt{1 - 24c\alpha_2(\alpha k - \alpha_1)} \right)$$

By choice, y_k satisfies $f'_k(y_k) = 0$. It might be helpful to keep in mind that

$$|24c\alpha_2(\alpha k - \alpha_1)| \leq \frac{48}{M\Lambda}$$

In particular, $24c\alpha_2(\alpha k - \alpha_1)$ is “relatively small.” The integral (44) can be written as

$$\frac{\exp(\operatorname{sgn}(\alpha_2)\frac{i\pi}{4})}{\sqrt{2|\alpha_2|}N^j} \exp(2\pi i f_k(y_k)) \int_0^N y^j \exp(2\pi i h_k(y - y_k)) dy$$

where

$$h_k(y) = h_k(y, c, \alpha_2) := cy^3 + \left(3cy_k - \frac{1}{4\alpha_2} \right) y^2$$

Some algebraic manipulations give

$$\begin{aligned} f_k(y_k) &= \frac{1}{864c^2\alpha_2^3} \left((1 - 24c\alpha_2(\alpha k - \alpha_1))^{3/2} - 1 + 36c\alpha_2(\alpha k - \alpha_1) \right. \\ &\quad \left. - 216c^2\alpha_2^2(\alpha k - \alpha_1)^2 \right) \\ &= \frac{1}{864c^2\alpha_2^3} \sum_{l=3}^{\infty} \frac{\prod_{r=1}^l (2r-5)}{2^l l!} 24^l c^l \alpha_2^l (\alpha k - \alpha_1)^l \\ &= \sum_{l=0}^{\infty} (24)^l g_l c^{l-2} \alpha_2^{l-3} \sum_{s=0}^l \binom{l}{s} (-1)^{l-s} \alpha_1^{l-s} \alpha^s k^s \\ &= \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} g_{l+s} (24)^{l+s} c^{l+s-2} \alpha_2^{l+s-3} \binom{l+s}{s} (-1)^l \alpha_1^l \alpha^s k^s \\ &= \sum_{s=0}^{\infty} d_s k^s \end{aligned} \tag{45}$$

where

$$\begin{aligned} d_s &:= (24)^s c^{s-2} \alpha_2^{s-3} \alpha^s q_s, & q_s &= \sum_{l=0}^{\infty} \binom{l+s}{s} g_{l+s} (-1)^l (24)^l c^l \alpha_2^l \alpha_1^l \\ g_0 &= 0, & g_1 &= 0, & g_2 &= 0, & |g_l| &\leq 1 \quad \text{for } l \geq 3 \end{aligned} \quad (46)$$

Note that g_l depends only on l . Since $|q_s| \leq 2^{s+1}$, $|\alpha_2| \leq \alpha$, $cN^2 \leq 1/\Lambda$, and $\alpha k \leq N$, it follows

$$|d_{s+3} k^{s+3}| \leq 2(48)^3 c \alpha^3 k^3 (48)^s c^s \alpha^{2s} M^s < \frac{N}{4M^s} \quad \text{for } s \geq 0 \quad (47)$$

Define

$$I_{k,j} := \frac{1}{\sqrt{2|\alpha_2|} N^j} \int_0^N y^j \exp(2\pi i h_k(y - y_k)) dy$$

Using the notation defined so far, the sum (43) has been reduced to a sum of the form

$$\sum_{k=T}^{M-T} \exp(2\pi i a k + 2\pi i b k^2 + 2\pi i f_k(y_k)) I_{k,j} \quad (48)$$

By Lemma 5.7, the computation of the sum (48) can be reduced to evaluating sums of the form

$$\frac{1}{M^r} \sum_{k=0}^M k^r \exp(2\pi i a k + 2\pi i b k^2 + 2\pi i f_k(y_k)) \quad (49)$$

where $0 \leq r = O(\Lambda)$ is an integer. By estimate (47), we can truncate the sum indexed by s in (45) at $S := 3 + \lfloor \log N / \log M \rfloor$. The remainder term $\sum_{s=S+1}^{\infty} d_s k^s$, which is bounded by $O(1)$, can then be routinely eliminated via Taylor expansions; see Section 2 for a similar calculation. This results in sums of the form

$$\frac{1}{M^t} \sum_{k=0}^M k^l \exp(2\pi i \beta_1 k + 2\pi i \beta_2 k^2 + \dots + 2\pi i \beta_S k^S) \quad (50)$$

where β_s are real numbers, $S = 3 + \lfloor \log N / \log M \rfloor$, and for $s \geq 0$

$$\begin{aligned} 0 &\leq l = O(\Lambda^2), & \Lambda &\leq M \leq \Lambda N \leq \Lambda^2 K^{2/5} \\ |\beta_1| &\in [0, 1/2], & |\beta_2| &\in [0, 1/2], & |\beta_{s+3}| &\leq \frac{N}{4M^{2s+3}} \end{aligned} \quad (51)$$

The sum (50) can be viewed in an alternative way. This viewpoint will be useful in Section 4.5 when M is small relative to N . Recall that $\beta_s = d_s$ for

$s \geq 3$, where d_s is defined as in (46). So, for $s \geq 1$, the coefficients β_{s+3} have the form

$$\begin{aligned}\beta_{s+3} &= \mu\eta_s, & \eta_s &:= \lambda^s \sum_{l=0}^{\infty} z_{l,s} \gamma^l \\ |z_{l,s}| &\leq 1/2, & \mu &:= 2(48)^3 c\alpha^3, & \lambda &:= 2c\alpha\alpha_2, & \gamma &:= 48c\alpha_1\alpha_2\end{aligned}$$

and $z_{l,s}$ depend only on l and s . Therefore, the sum (50) can be presented as

$$\frac{1}{M^l} \sum_{k=0}^M k^l \exp(2\pi i \beta_1 k + 2\pi i \beta_2 k^2 + 2\pi i \beta_3 k^3 + 2\pi i \mu \eta_1 k^4 + \dots + 2\pi i \mu \eta_{S-3} k^S) \quad (52)$$

and for $s \geq 1$ we have

$$|\mu| \leq \frac{N}{M^3}, \quad |\lambda| \leq \frac{1}{M^2}, \quad |\gamma| \leq \frac{1}{M^2}, \quad |\mu\eta_s| \leq \frac{N}{4M^{2s+3}} \quad (53)$$

In particular, if μ , λ , and γ are determined, then so are the coefficients $\mu\eta_s$.

4.4 The third loop: the FFT precomputation

We want to precompute sums of the form (50) with conditions (51). So, let n be the largest integer satisfying $2^n \leq M$ and let R be the smallest integer that satisfies $2^R > \Lambda^2 K^{2/5}$. Note that $n < R$, and R depends only on K and Λ . We may assume that $n > 0$.

Instead of evaluating (50), let us evaluate the more general sum

$$F_q(M, \tilde{M}; \vec{\beta}) := \frac{1}{M^l} \sum_{k=0}^{\tilde{M}} (q+k)^l \exp(2\pi i \beta_1 k + \dots + 2\pi i \beta_{\tilde{S}} k^{\tilde{S}})$$

where $q, \tilde{M} \geq 0$ are integers, $q + \tilde{M} \leq M$, $\vec{\beta} := (\beta_1, \dots, \beta_{\tilde{S}})$, and

$$\tilde{S} = \lfloor 3 + R/n \rfloor, \quad |\beta_s| \leq \min \left\{ 1/2, 2^{R-2} 2^{n(3-2s)} \right\}$$

To this end, let $q_0 := 0$, $M_0 := M$, $n_0 := n$, and $\vec{\beta}^{(0)} := \vec{\beta}$. For integers $d \geq 0$, and for as long as $M_d \geq 1$, sequentially define

$$\begin{aligned}M_{d+1} &:= M_d - 2^{n_d}, & n_{d+1} &:= \text{the largest integer satisfying } 1 \leq 2^{n_{d+1}} \leq M_{d+1} \\ q_{d+1} &:= q_d + 2^{n_d}, & \vec{\beta}^{(d+1)} &:= (\beta_1^{(d+1)}, \dots, \beta_{\tilde{S}}^{(d+1)}), & \beta_s^{(d+1)} &= \sum_{p=s}^{\tilde{S}} \binom{p}{s} 2^{n_d(p-s)} \beta_p^{(d)}\end{aligned}$$

Note $M_d + q_d \leq M$, $M_{d+1} < M_d/2$, and $n_d \leq n - d$. With this notation, we have

$$F_{q_d}(M, M_d; \vec{\beta}^{(d)}) = F_{q_d}(M, 2^{n_d} - 1; \vec{\beta}^{(d)}) + c_d F_{q_{d+1}}(M, M_{d+1}; \vec{\beta}^{(d+1)}) \quad (54)$$

where c_d is computable in $O(\tilde{S}^2)$ operations and satisfies $|c_d| = 1$.

We want to obtain a bound on the coefficients $\beta_s^{(d)}$. By induction, suppose the inequality

$$|\beta_s^{(d)}| \leq 2^{R-2} 2^{n(3-2s)} 2^{sd} (1 + 1/(2R))^d \quad (55)$$

holds for $3 \leq s \leq \tilde{S}$. Note condition (55) is satisfied for $d = 0$ and $3 \leq s \leq \tilde{S}$. For $n \geq \log_2 R + 4$ we have

$$\begin{aligned} |\beta_s^{(d+1)}| &\leq 2^{R-2} 2^{n(3-2s)} 2^{s(d+1)} (1 + 1/(2R))^d \left| \sum_{p=0}^{\tilde{S}-s} 2^p 2^{pn_d} 2^{-2pn} 2^{pd} \right| \\ &\leq 2^{R-2} 2^{n(3-2s)} 2^{s(d+1)} (1 + 1/(2R))^{d+1} \end{aligned}$$

So, estimate (55) holds for such n . If $n < \log_2 R + 4$, then $M \leq 32R$, and the sum can be evaluated directly.

Using Taylor expansions, the evaluation of $F_{q_d}(M, 2^{n_d} - 1; \vec{\beta}^{(d)})$ is hence reduced to computing sums of the form

$$\frac{1}{2^{un_d}} \sum_{k=0}^{2^{n_d}-1} k^u \exp \left(2\pi i \frac{\tilde{\sigma}_{n,1}^{(d)}}{2^{n_d}} k + 2\pi i \frac{\tilde{\sigma}_{n,2}^{(d)}}{2^{2n_d}} k^2 + \dots + 2\pi i \frac{\tilde{\sigma}_{n,\tilde{S}}^{(d)}}{2^{\tilde{S}n_d}} k^{\tilde{S}} \right)$$

where $0 \leq u = O(\Lambda^3)$ is an integer, and by estimate (55), the integers $\tilde{\sigma}_{n,s}^{(d)}$ satisfy

$$|\tilde{\sigma}_{n,1}^{(d)}| \leq 2^{n_d-1}, \quad |\tilde{\sigma}_{n,2}^{(d)}| \leq 2^{2n_d-1}, \quad |\tilde{\sigma}_{n,s}^{(d)}| \leq \min \left\{ 2^{sn_d-1}, 2^{R-1} 2^{(3-s)n} \right\}$$

provided $\log_2 R + 4 \leq n < R$. Finally, to evaluate $F_{q_{d+1}}(M, M_{d+1}; \vec{\beta}^{(d+1)})$, simply iterate the relation (54).

In summary, after at most $R = O(\log K)$ iterations of relation (54), our problem is reduced to evaluating the sums

$$\frac{1}{2^{l(n-d)}} \sum_{k=0}^{2^{(n-d)}-1} k^l \exp \left(2\pi i \frac{\sigma_{n,1}^{(d)}}{2^{(n-d)}} k + \dots + 2\pi i \frac{\sigma_{n,\tilde{S}}^{(d)}}{2^{\tilde{S}(n-d)}} k^{\tilde{S}} \right) \quad (56)$$

with conditions

$$\begin{aligned} 2^R &\leq 2\Lambda^2 K^{2/5}, & 1 &\leq n \leq R-1, & \tilde{S} &= \lfloor 3 + R/n \rfloor \\ 0 &\leq l = O(\Lambda^3), & 0 &\leq d \leq n-1, & |\sigma_{n,1}^{(d)}| &\leq 2^{(n-d)-1} \\ |\sigma_{n,2}^{(d)}| &\leq 2^{2(n-d)-1}, & |\sigma_{n,s}^{(d)}| &\leq \min \left\{ 2^{s(n-d)-1}, 2^{R-1} 2^{(3-s)n} \right\} \end{aligned} \quad (57)$$

4.5 The cost of the FFT precomputation

We use FFT to precompute the sums (56) with conditions (57). Note that for $0 \leq s \leq \lfloor R/n + 3 \rfloor$, we have $R + 3n - sn \geq 0$. So, for a given n , d , and l , the

FFT cost is at most

$$\prod_{s=0}^{\lfloor 3+R/n \rfloor} \min \{2^{sn}, 2^{R+3n-sn}\} \quad (58)$$

But $\min \{2^{sn}, 2^{R+3n-sn}\} = 2^{sn}$ for $s < \lceil R/2n + 3/2 \rceil$. A straightforward calculation then reveals the quantity (58) is bounded by

$$2^{(\lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil + 1)(R/n+3 - \lceil R/2n+3/2 \rceil)n} \times \prod_{s=0}^{\lceil R/2n+3/2 \rceil - 1} 2^{sn} \prod_{s=0}^{\lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil} 2^{-sn} \quad (59)$$

Note that $0 \leq \lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil$, so neither of the products above is empty.

Define $f(n) := n(R/2n+3/2)^2$. By Lemma 5.8, the quantity (59) is at most $2^{f(n)}$. But $f(n) \leq 4R$ exactly when $R/9 \leq n \leq R$. So, for each $R/9 \leq n \leq R$, $0 \leq d < n-1$, and $0 \leq l = O(\Lambda^3)$ the cost of the FFT precomputation is at most 2^{4R} . Therefore, the total cost for all $R/9 \leq n \leq R$, $0 \leq d < n-1$, and $0 \leq l = O(\Lambda^3)$ is less than $\Lambda^9 K^{8/5}$ operations.

If $n < R/9$, then the length of the remaining sum is at most $2^{R/9} \leq \Lambda K^{2/45}$. It is possible to directly evaluate such sums. However, if we do so, a simple optimization procedure then shows that the complexity of the resulting algorithm to compute the zeta function will be only $t^{37/117+o(1)}$. We want to avoid direct computation of these sums so that we are able to achieve $t^{4/13+o(1)}$ complexity. This is where the alternative viewpoint (52) with conditions (53), which was discussed in Section 4.3, becomes helpful.

According to formulation (52), the sums to be evaluated have the form

$$\frac{1}{M^u} \sum_{k=0}^M k^u \exp(2\pi i \beta_1 k + 2\pi i \beta_2 k^2 + \dots + 2\pi i \beta_S k^S)$$

where $\Lambda < M \leq 2^{R/9}$, $0 \leq u = O(\Lambda^2)$ an integer, $S = \lfloor 3 + \log N / \log M \rfloor$, and for integers $1 \leq s \leq S-3$

$$\beta_{s+3} = \mu \eta_s, \quad \eta_s := \lambda^s \sum_{l=0}^{\infty} z_{l,s} \gamma^l$$

Recall that the coefficients $z_{l,s}$ depend only on l and s . They also satisfy the uniform bound $|z_{l,s}| \leq 1/2$. According to set of conditions (53), the parameters μ , λ , and γ conform to the bounds

$$|\mu| \leq 2^{R-3\rho}, \quad |\lambda| \leq 2^{-2\rho}, \quad |\gamma| \leq 2^{-2\rho}, \quad |\mu \eta_s| \leq 2^{R-3\rho-2s\rho}$$

where $\rho := \log_2 M$. By hypothesis, $\rho \leq R/9 + 1$.

We perform ‘‘perturbation analysis’’ on β_{s+3} for $s \geq 1$. To this end, let ϵ_1 be a real number satisfying $|\epsilon_1| \leq 2^{-2\rho}$. Then

$$\left| (\mu + \epsilon_1) \lambda^s \sum_{l=0}^{\infty} z_{l,s} \gamma^l \right| 2^{3\rho+s\rho} = \mu \eta_s 2^{3\rho+s\rho} + O(\epsilon_1 2^{(3-s)\rho})$$

Since $s \geq 1$, Taylor expansions allow us to discretize μ using step size $2^{-2\rho}$. Since $|\mu| \leq 2^{R-3\rho}$, this yields $2^{R-\rho}$ possible discretizations for μ . Similarly

$$\left| \mu(\lambda + \epsilon_1)^s \sum_{l=0}^{\infty} z_{l,s} \gamma^l \right| 2^{3\rho+s\rho} = \mu \eta_s 2^{3\rho+s\rho} + O(\epsilon_1 2^s 2^{R+2\rho-s\rho})$$

Therefore, we need to determine λ only up to a multiple of $2^{-R-\rho}$ (note this is less than ϵ_1). Since $|\lambda| \leq 2^{-2\rho}$, this gives $2^{R-\rho}$ discretized values of λ . Finally

$$\left| \mu \lambda^s \sum_{l=0}^{\infty} z_{l,s} (\gamma + \epsilon_1)^l \right| 2^{3\rho+s\rho} = \mu \eta_s 2^{3\rho+s\rho} + O(\epsilon_1 2^{R-s\rho})$$

Thus, it suffices to dissect γ using $2^{\rho-R}$ step size. Since $|\gamma| \leq 2^{-2\rho}$, there are at most $2^{R-3\rho}$ possible discretizations of γ .

To conclude, for each u , S , and M the cost of FFT is $O(2^{6\rho} 2^{3R-5\rho}) = O(2^{3R+\rho})$. Since $\rho \leq R/9 + 1$, there are $O(\Lambda^2 R 2^{R/9})$ possible tuples (u, S, M) . It follows that the total cost is $O(\Lambda^2 R 2^{3R+2R/9}) = O(\Lambda^9 K^{8/5})$.

5 Auxiliary results

Suppose $K > 0$, $j \geq 0$, and $M \geq 0$ are integers, and c is a real number. Let B_{2m} denote the Bernoulli numbers. Recall that $B_{2m} = O((2\pi)^{-2m} (2m)!)$. Let $f_{x,c,j}(y)$ and E_M denote a function and a quantity satisfying

$$f_{x,c,j}(y) = \frac{y^j}{K^j} \exp(2\pi i c y^3 + 2\pi i x y), \quad |E_M| \leq \frac{2}{(2\pi)^{2M}} \int_0^K \left| f_{x,c,j}^{(2M+1)}(y) \right| dy$$

Lemma 5.1. *For any $j \geq 0$, any $K > \Lambda(K, j, \epsilon)$, and any real c , Euler-Maclaurin summation gives*

$$\begin{aligned} \frac{1}{K^j} \sum_{n=0}^K n^j \exp(2\pi i c n^3 + 2\pi i x n) &= \frac{1}{K^j} \int_0^K y^j \exp(2\pi i c y^3 + 2\pi i x y) dy + \\ \frac{1}{2} (f_{x,c,j}(K) + f_{x,c,j}(0)) &+ \sum_{m=1}^M \frac{B_{2m}}{(2m)!} \left(f_{x,c,j}^{(2m-1)}(K) - f_{x,c,j}^{(2m-1)}(0) \right) + E_M \end{aligned}$$

Lemma 5.2. *For any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any positive integer K such that $K > \Lambda(K, j, \epsilon)$, any $x \in [-3/4, 3/4]$, and any real c satisfying $|c|K^2 \leq 1/48$, we have*

$$\max_{0 \leq y \leq K} \left| f_{x,c,j}^{(m)}(y) \right| \leq (6\pi c K^2 + 3\pi/2 + (2m+j)/K)^m$$

In particular, if $M = \lceil 8 \log K/\epsilon \rceil$, then $|E_M| < \epsilon$.

Proof. We can write $f_{x,c,j}^{(m)}(y) = P_{x,c,m,j}(y) \exp(2\pi icy^3 + 2\pi ixy)$, where $P_{x,c,m,j}(y)$ is a polynomial in y of degree $2m+j$ (and a polynomial in x of degree m). In particular, $P_{x,c,m,j}(y)$ has the form $\sum_{l=0}^{2m+j} d_{l,x,c}^{m,j} y^l$, where $d_{l,x,c}^{m,j}$ are polynomials in x of degree at most m . Let $|\sum_{l < m} z_l^t|_{1,t} := \sum_{l < m} |z_l| |t|^l$ denote an l^1 -norm. For $z_l(x)$ polynomials in x , define $|\sum_{l < m} z_l(x) y^l|_{1,y,x} := \sum_{l < m} |z_l(x)|_{1,x} |y|^l$. By induction, suppose that

$$\max_{0 \leq y \leq K} |P_{x,c,m,j}(y)|_{1,x,y} \leq (6\pi cK^2 + 3\pi/2 + (2m+j)/K)^m$$

Then

$$\max_{0 \leq y \leq K} |P'_{x,c,m,j}(y)|_{1,x,y} \leq K^{-1}(2m+j)(6\pi cK^2 + 3\pi/2 + (2m+j)/K)^m$$

It follows that

$$\begin{aligned} \max_{0 \leq y \leq K} |P_{x,c,m+1,j}(y)|_{1,x,y} &\leq \max_{0 \leq y \leq K} |6\pi cy^2 P_{x,c,m,j}(y)|_{1,x,y} + \\ &\quad \max_{0 \leq y \leq K} |2\pi x P_{x,c,m,j}(y)|_{1,x,y} + \max_{0 \leq y \leq K} |P'_{x,c,m,j}(y)|_{1,x,y} \\ &\leq (6\pi cK^2 + 3\pi/2 + (2m+2+j)/K)^m \end{aligned}$$

□

Lemma 5.3. *There exist absolute constants κ_7, κ_8 such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any positive integers K, K_1 satisfying $\Lambda(K, j, \epsilon) \leq K_1 \leq K$, any $a, b \in \mathbb{R}/\mathbb{Z}$, any real c satisfying $|cK_1^2| < 1/48$, any $1 \leq m = O(\nu(K, j, \epsilon))$, any $\alpha \in [-1, 1]$, and any interval $[w, z] \subset [-1, 1]$, the sum*

$$\frac{B_{2m}}{(2m)!} \int_w^z \left(f_{x,c_1,j}^{(2m-1)}(K_1) - f_{x,c_1,j}^{(2m-1)}(0) \right) F(K; a - \alpha x, b) dx \quad (60)$$

can be computed with error bounded by $O(\nu(K, j, \epsilon)^{\kappa_7} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\kappa_8})$ arithmetic operations. The big- O constants are absolute and arithmetic is performed using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. Write $f_{x,c,j}^{(m)}(y) = P_{y,c,m,j}(x) \exp(2\pi icy^3 + 2\pi ixy)$. Then $P_{y,c,m,j}(x)$ has the form $\sum_{l=0}^m z_{l,y,c}^{m,j} x^l$, where $z_{l,y,c}^{m,j}$ are polynomials in y of degree $\leq 2m+j$. From the proof of Lemma 5.2, $|z_{l,y,c}^{m,j}|_{1,x} \leq (6\pi cK_1^2 + 3\pi/2 + (2m+j)/K_1)^m = O((2\pi)^{2m})$. So, in order to compute (60), it is enough to compute sums of the form

$$\int_w^z x^l \exp(2\pi i K_1 x) F(K; a - \alpha x, b) dx \quad \int_w^z x^l F(K; a - \alpha x, b) dx \quad (61)$$

The sums (61) are handled by Lemma 5.10. □

Lemma 5.4. *There exist absolute constants κ_9, κ_{10} such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any positive integers K, K_1 satisfying $\Lambda(K, j, \epsilon) \leq K_1 \leq K$,*

any $a, b \in \mathbb{R}/\mathbb{Z}$, any real c satisfying $|cK_1^2| < 1/48$, any $\alpha \in [-1, 1]$, and any interval $(w, z) \subset (-1, 1)$ such that $|w| \geq 1/4$, the function

$$\frac{1}{K_1^j} \int_w^z \int_0^{K_1} y^j e^{2\pi i c y^3 - 2\pi i x y} F(K; a + \alpha x, b) dy dx \quad (62)$$

can be computed with error bounded by $O(\nu(K, j, \epsilon)^{\kappa_9} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\kappa_{10}})$ arithmetic operations. The big- O constants are absolute and arithmetic is performed using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. Conjugating if necessary, we may assume that w is positive. Define the contours

$$C_1 := \{te^{-i\pi/6} | 0 \leq t \leq 2K_1/\sqrt{3}\}, \quad C_2 := \{K_1 - it | 0 \leq t \leq K_1/\sqrt{3}\}$$

as well as $C_0 := \{t | 0 \leq t \leq K_1\}$. Also define

$$I_x(C) := \frac{1}{K_1^j} \int_C y^j e^{2\pi i c y^3 - 2\pi i x y} dy$$

With this notation, (62) can be rewritten as $\int_w^z I_x(C_0) F(K; a + \alpha x, b) dx$. By Cauchy's Theorem $I_x(C_0) = I_x(C_1) - I_x(C_2)$. But

$$\begin{aligned} I_x(C_1) &= \frac{d_1}{K_1^j} \int_0^{\frac{2K_1}{\sqrt{3}}} y^j e^{2\pi i c y^3 - \sqrt{3}\pi i x y - \pi x y} dy \\ I_x(C_2) &= \frac{d_2 e^{-2\pi i K_1 x}}{K_1^j} \int_0^{\frac{K_1}{\sqrt{3}}} (K_1 - iy)^j e^{6\pi i c K_1^2 y - 6\pi i c K_1 y^2 - 2\pi i c y^3 - 2\pi i x y} dy \end{aligned}$$

where the constants d_1, d_2 have modulus one. Since $x \geq w \geq 1/4$, $cK_1^2 \leq 1/48$, the moduli of the integrands in both $I_x(C_1)$ and $I_x(C_2)$ decline in the region $[0, \sqrt{2}K_1]$ faster than $e^{-x/6}$. So, in both cases we can truncate the domain of integration with respect to y at $L = \nu(K, j, \epsilon)$. Once truncated, the integrals with respect to y can be subdivided into L consecutive subintegrals. Taylor expansions can then be used to reduce to integrals of the form

$$\begin{aligned} &\int_w^z \int_0^1 y^l x^s e^{-\sqrt{3}\pi i n x - \pi n x} F(K; a + \alpha x, b) dy dx \\ &\int_w^z \int_0^1 y^l x^s e^{-2\pi i K_1 x - 2\pi n x} F(K; a + \alpha x, b) dy dx \end{aligned}$$

where $0 \leq n < L$, $0 \leq s = O(\nu(K, j, \epsilon))$, and $0 \leq l = O(\nu(K, j, \epsilon))$. We integrate directly with respect to y . As for the integral with respect to x , make the change of variable $nx \rightarrow x$ if $n > 0$. Then, subdivide the resulting integral into n consecutive subintegrals. Finally, use Taylor expansions to reduce to integrals of the form (61), which are handled by Lemma 5.10. \square

Lemma 5.5. *There exist absolute constants κ_{11}, κ_{12} such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any positive integers K and K_1 satisfying $\Lambda(K, j, \epsilon) < K_1 < K$, any integer $1 \leq m = O(K_1^3)$, any integer $0 \leq p = O(\nu(K, j, \epsilon))$, any $\beta \in [-m, m]$, any $\alpha \in [-m, m]$, any $\lambda \in [0, m]$, any $\theta \in \{-1, 1\}$, any $\eta \in [-m, m]$, any interval $(0, w)$ where $0 < w = O(1)$, and any real c satisfying $|cK_1^2| < 1/\Lambda(K, j, \epsilon)$, the integral*

$$\frac{1}{K_1^j} \int_0^{K_1} y^j e^{2\pi i c y^3 + 2\pi i \eta y} \int_0^w x^p e^{2\pi i (\alpha - \theta y)x - 2\pi \lambda x - 2\pi i \beta x^2} dx dy \quad (63)$$

can be computed with error bounded by $O(\nu(K, j, \epsilon)^{\kappa_{11}} K^{-2}\epsilon)$ using $O(\nu(K, j, \epsilon)^{\kappa_{12}})$ arithmetic operations. The big- O constants are absolute and arithmetic is done using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. Let us first assume $\lambda = 0$. Conjugating if necessary, we may also assume β is a non-negative number. We comment it is possible to replace the assumption $\theta \in \{-1, 1\}$ with $\theta \in [-1, 1]$ say. We do not pursue this as the Lemma is already more general than is needed for the purposes of this paper.

Define $L := 8(p+1)^2 \nu(K, j, \epsilon)$. Suppose $\beta \leq L$. Then, make the change of variable $x \rightarrow Lx$ in (63). Subdivide the resulting integral into $O(L)$ consecutive subintegrals over the intervals $[n, n + \delta]$, where $0 \leq n = O(L)$ an integer and $\delta \in (0, 1]$. Note δ is usually 1 except possibly for the last interval. For each subintegral, make the change of variable $x \rightarrow x - n$. Then use Taylor expansions to reduce the problem to evaluating integrals of the form

$$\frac{1}{K_1^j} \int_0^{K_1} y^j e^{2\pi i c y^3 + 2\pi i \eta_1 y} \int_0^\delta x^s e^{2\pi i \frac{\alpha_1 - \theta y}{L} x} dx dy \quad (64)$$

where $0 \leq s = O(\nu(K, j, \epsilon))$ an integer, and $\eta_1, \alpha_1 = O(K_1^3)$ real numbers. Split the integral with respect to y into two subintegrals over the intervals

$$I_1 := \{y \in [0, K_1] \mid |y - \theta\alpha_1| \geq 2(s+1)L\}, \quad I_2 := [0, K_1] \setminus I_1$$

If I_2 is not empty, make the change of variable $\alpha_1 - \theta y \rightarrow y$, then eliminate the resulting cross term $e^{2\pi i xy/L}$ using appropriate changes of variable and Taylor expansions. Note since I_2 is not empty, then $|\alpha_1| \leq K_1 + 2(s+1)L$. So, this procedure results in integrals of the form (69); see below. These integrals can be evaluated efficiently.

As for the interval I_1 , integrate directly with respect to x in (64) to obtain

$$\frac{s! L^r}{(s+1-r)! K_1^j} \int_{I_1} y^j (\theta\alpha_1 - y)^{-r} e^{2\pi i c y^3 + 2\pi i \eta_2 y} dy \quad (65)$$

where $1 \leq r \leq s+1$ an integer and $\eta_2 = O(\eta_1 + 1)$ a real number. Since $|y - \theta\alpha_1| \geq 2(s+1)L$ for all $y \in I_1$, it is not hard to see that we can subdivide (65) into $O(\log K_1)$ subintegrals of the form

$$\frac{s! L^r}{(s+1-r)! K_1^j} \int_A^{A+\Delta} y^j (\theta\alpha_1 - y)^{-r} e^{2\pi i c y^3 + 2\pi i \eta_2 y} dy \quad (66)$$

where $A \in I_1$, $\Delta \in (0, A]$, $[A, A + \Delta] \subset I_1$, and $|\Delta/\tilde{A}| \leq 1/2$, $\tilde{A} := \theta\alpha_1 - A$. Note $|\tilde{A}| \geq 2(s+1)L$. Make the change of variable $y \rightarrow y - A$ in (66) to obtain an integral of the form

$$\frac{s! L^r}{(s+1-r)! K_1^j} \int_0^\Delta (A+y)^j (\tilde{A}-y)^{-r} e^{2\pi i c(y+A)^3 + 2\pi i \eta_2 y} dy \quad (67)$$

Write $(\tilde{A}-y)^{-r}$ as $\tilde{A}^{-r}(1-y/\tilde{A})^{-r}$, then apply Taylor expansions to the term $(1-y/\tilde{A})^{-r}$. This reduces the integral (67) to the form

$$\frac{s! L^r 2^{r+h}}{(s+1-r)! K_1^j \tilde{A}^{r+h}} \int_0^\Delta (A+y)^j y^h e^{2\pi i c(y+A)^3 + 2\pi i \eta_2 y} dy \quad (68)$$

where $0 \leq h = O(\nu(K, j, \epsilon))$ an integer. Expand the term $(A+y)^j$ as well as the exponent $(y+A)^3$ of $e^{2\pi i c(y+A)^3}$. The problem then reduces to evaluating integrals of the form

$$\frac{1}{\Delta^u} \int_0^\Delta y^u e^{2\pi i c y^3 + 2\pi i \eta_4 y^2 + 2\pi i \eta_3 y - 2\pi \rho y} dy \quad (69)$$

where $0 \leq u = O(\nu(K, j, \epsilon))$ an integer, $\Delta \leq K_1$, $0 \leq \rho$, $\eta_3 = O(\eta_2 + 1)$ a real number, and $\eta_4 = O(1)$ a real number.

The integral (69) is a variation on the Airy integral; see [GK] for example. We show how to compute it efficiently in Section 4.3 and Lemma 5.7. In fact, this integral can still be computed efficiently even if we relax some of the conditions on c , η_3 , and η_4 . Let us elaborate. If $\rho = 0$ in (69), then consider two cases: either the integrand has a critical point or it does not; that is, either $3cy^2 + 2\eta_4 y + \eta_3$ has a zero in $[0, \Delta]$ or it does not. In the former case, extract the saddle-point contribution as done in Section 4.3. In the latter case, use Cauchy's theorem to shift integration contours to the stationary phase of the integrand as done in Lemma 5.7. If $\rho > 0$ in (69), then the integral can still be computed efficiently; we only need to truncate it where appropriate, then apply Taylor expansions to obtain either integrals of the form (69) but with $\rho = 0$ or other elementary and easily-calculable integrals.

We may now assume $\beta \geq L$. For simplicity, we will also assume $p = 0$, which is in fact the only case needed for the purposes of this paper. Let $I_4 \subset [0, K_1]$ denote the interval satisfying $(\alpha - \theta y)/(2\beta) \in [0, w]$ for $y \in I_4$, and $I_5 := [0, K_1] \setminus I_4$. We wish to evaluate

$$\frac{1}{K_1^j} \int_0^{K_1} y^j e^{2\pi i c y^3 + 2\pi i \eta y} \int_0^w e^{2\pi i (\alpha - \theta y)x - 2\pi i \beta x^2} dx dy \quad (70)$$

Let us do that over the domain $(x, y) \in [0, w] \times I_5$ first. We may write $I_5 = I_6 \cup I_7$, where I_6 is the interval where $\alpha - \theta y > 2\beta w$ and I_7 is the interval where $\alpha - \theta y < 0$. We evaluate (70) when $y \in I_6$ first. Assume I_6 is not empty. Let $\epsilon_1 := K^{-2}\epsilon$. Write $I_6 = I_{\epsilon_1} \cup \tilde{I}_6$, where \tilde{I}_6 is the subinterval of I_6 where $\alpha - y \geq 2\beta w + \epsilon_1$ and $I_{\epsilon_1} := I_6 \setminus \tilde{I}_6$. Assume \tilde{I}_6 is not empty.

Then, Cauchy's theorem followed by the change of variable $\sqrt{2\beta}x \rightarrow x$ reduce the integral (70) when $x \in [0, w]$ and $y \in \tilde{I}_6$ to integrals of the form

$$\begin{aligned} & \frac{1}{K_1^j} \int_{\tilde{I}_6} y^j e^{2\pi i c y^3 + 2\pi i \eta y} \frac{1}{\sqrt{2\beta}} \int_0^\infty e^{-2\pi \tau_1(y)x + \pi i x^2} dx dy \\ & \frac{1}{K_1^j} \int_{\tilde{I}_6} y^j e^{2\pi i c y^3 + 2\pi i \tilde{\eta} y} \frac{1}{\sqrt{2\beta}} \int_0^\infty e^{-2\pi \tau_2(y)x + \pi i x^2} dx dy \end{aligned} \quad (71)$$

where $\tilde{\eta} = O(\eta + 1)$ a real number, and

$$\tau_1(y) := \frac{\alpha - \theta y}{\sqrt{2\beta}}, \quad \tau_2(y) := \frac{\alpha - \theta y - 2\beta w}{\sqrt{2\beta}}$$

We evaluate the second integral in (71), the first integral being even easier to evaluate. Note that $\tau_2(y) \geq \epsilon_1/\sqrt{2\beta} > 0$ for all y in the closure of \tilde{I}_6 . So by Cauchy's theorem, the inner integral in the second integral in (71) can be replaced with

$$\frac{1}{\sqrt{2\beta}} \int_0^\infty e^{-2\pi e^{\pi i/4} \tau_2(y)x - \pi x^2} dx \quad (72)$$

Let $R := \lceil \sqrt{8\nu(K, j, \epsilon)} \rceil$. Note we can truncate the integral (72) at R . The truncated integral can then be subdivided into R consecutive subintegrals over intervals $[n, n+1]$, where $0 \leq n < R$ an integer. For each subintegral, make the change of variable $x \rightarrow x - n$, then apply Taylor expansions to quadratic factor in the integrand. This yields integrals of the form

$$\frac{e^{-2\pi e^{\pi i/4} \tau_2(y)n}}{\sqrt{2\beta}} \int_0^1 x^q e^{-2\pi e^{\pi i/4} \tau_2(y)x - 2\pi n x} dx \quad (73)$$

where $0 \leq n < R$ and $0 \leq q = \nu(K, j, \epsilon)$ integers. Plug (73) back into the second integral in (71). We obtain

$$\frac{1}{\sqrt{2\beta} K_1^j} \int_{\tilde{I}_6} y^j e^{2\pi i c y^3 + 2\pi i \tilde{\eta} y - 2\pi e^{\pi i/4} \tau_2(y)n} \int_0^1 x^q e^{-2\pi e^{\pi i/4} \tau_2(y)x - 2\pi n x} dx dy \quad (74)$$

Split \tilde{I}_6 into two intervals \tilde{I}_7 and \tilde{I}_8 , where \tilde{I}_7 is the interval that satisfies $|e^{\pi i/4} \tau_2(y) + n| \leq q$ and $\tilde{I}_8 := \tilde{I}_6 \setminus \tilde{I}_7$. Note \tilde{I}_7 and \tilde{I}_8 , as do all other intervals in this proof, consist of $O(1)$ many intervals of the form $[s, t]$ where $s, t \in [0, K_1]$.

Over \tilde{I}_7 make the change of variable $x \rightarrow qx$, if $q > 0$, and subdivide the resulting integral into q consecutive subintegrals over the intervals $[m, m+1]$, where $0 \leq m < q$ an integer. For each subintegral, make the change of variable $x \rightarrow x - m$, then apply Taylor expansions to finally obtain integrals of the form

$$\frac{1}{\sqrt{2\beta} K_1^j} \int_{\tilde{I}_7} \frac{(e^{\pi i/4} \tau_2(y) + n)^h}{q^h} y^j e^{2\pi i c y^3 + 2\pi i \tilde{\eta} y - 2\pi e^{\pi i/4} \tau_2(y)(n+m/q)} \int_0^1 x^{q+h} dx dy \quad (75)$$

where $0 \leq h = O(\nu(K, j, \epsilon))$ an integer. To conclude, directly integrate with respect to x in (75), then note the remaining integral with respect to y can be easily written in terms of integrals of the form (69), which can be evaluated efficiently.

As for the interval \tilde{I}_8 , directly integrate with respect to x in (74). This yields integrals of the form

$$\frac{q!}{(q+1-r)! \sqrt{2\beta} K_1^j} \int_{\tilde{I}_6} \frac{y^j}{(e^{\pi i/4} \tau_2(y) + n)^r} e^{2\pi i c y^3 + 2\pi i \eta y - 2\pi e^{\pi i/4} \tau_2(y) n} dy \quad (76)$$

where $1 \leq r \leq q+1$ is an integer. Once again, the integral (76) can be routinely reduced to integrals of the form (69).

It remains to evaluate the integral (70) over the domain $(x, y) \in [0, w] \times I_{\epsilon_1}$. But the magnitude of (70) over this domain is $O(K^{-2}\epsilon)$. Therefore, we can ignore the domain $[0, w] \times I_{\epsilon_1}$. Also, it is not too hard to see that the evaluation of the integral (70) when the domain of integration is set to $[0, w] \times I_7$ is similar to the case $[0, w] \times I_6$ already considered.

Lastly, we evaluate (70) when the domain of integration is restricted to $(x, y) \in [0, w] \times I_4$. Note $\tau_1(y) \geq 0$ whereas $\tau_2(y) \leq 0$ over I_4 . An application of Cauchy's theorem followed by the change of variable $\sqrt{2\beta}x \rightarrow x$ reduce (70) over $[0, w] \times I_4$ to integrals of the form

$$\frac{1}{\sqrt{2\beta} K_1^j} \int_{I_4} y^j e^{2\pi i c y^3 + 2\pi i \eta y} \int_0^{2\sqrt{\beta}w} e^{2\pi e^{\pi i/4} \tau_1(y)x - \pi x^2} dx dy \quad (77)$$

$$\frac{1}{\sqrt{2\beta} K_1^j} \int_{I_4} y^j e^{2\pi i c y^3 + 2\pi i \eta y} \int_0^{\sqrt{2\beta}w} e^{2\pi \tau_2(y)x + \pi i x^2} dx dy \quad (78)$$

The integral (78) can be evaluated in a similar way to the second integral in (71). As for the integral (77), one can first compute it over the domains $[2\sqrt{\beta}w, \infty) \times I_4$ and $(-\infty, 0] \times I_4$. Over these domains, the integral with respect to x in (77) declines exponentially in both its linear and quadratic arguments. It is then not too hard to see that over these domains, the integral (77) can be reduced to the form (69); also see Lemma 5.6. So, instead of evaluating (77) over $[0, 2\sqrt{\beta}w] \times I_4$ we evaluate it over $(-\infty, \infty) \times I_4$. This yields

$$\begin{aligned} & \frac{1}{\sqrt{2\beta} K_1^j} \int_{I_4} y^j e^{2\pi i c y^3 + 2\pi i \eta y} \int_{-\infty}^{\infty} e^{2\pi e^{\pi i/4} \tau_1(y)x - \pi x^2} dx dy \\ &= \frac{1}{\sqrt{2\beta} K_1^j} \int_{I_4} y^j e^{2\pi i c y^3 + 2\pi i \eta y + \pi i \tau_1^2(y)} dy \end{aligned} \quad (79)$$

The integral (79) can be reduced to the form (69), which can be computed efficiently.

Finally, the treatment of (63) when $\lambda > 0$ is analogous. \square

Lemma 5.6. *There exist absolute constants κ_{13} and κ_{14} such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any positive integers K and K_1 satisfying $\Lambda(K, j, \epsilon) < K_1 < K$, any $\alpha \in [1/\Lambda(K, j, \epsilon), K_1]$, any $\alpha_1 \in [0, 4\alpha]$, any real number α_2 satisfying $1/K^2 \leq |\alpha_2| \leq \alpha$ and $\alpha_2/\alpha^2 = O(1)$, any $a, b \in \mathbb{R}/\mathbb{Z}$, any real number c_1 satisfying $|c_1 K_1^2| < 1/\Lambda(K, j, \epsilon)$, any positive integer K_2 satisfying, if possible, the bound $2\Lambda(K, j, \epsilon)^2 < K_2 \leq K_1/\alpha$, and for contours $C_0 = \{t | 1/4 < t < \infty\}$, $\overline{C_0} = \{t | -\infty < t < -1/4\}$, the sum*

$$\frac{1}{K_1^j} \sum_{k=\lfloor \Lambda^2 \rfloor}^{K_2 - \lfloor \Lambda^2 \rfloor} e^{2\pi i a k + 2\pi i b k^2} \int_0^{K_1} \int_C y^j e^{2\pi i c_1 y^3 - 2\pi i x y} e^{2\pi i (\alpha k - \alpha_1)x - 2\pi i \alpha_2 x^2} dx dy \quad (80)$$

where $C_0 \in \{C_0, \overline{C_0}\}$, can be computed with error bounded by $O(\nu(K, j, \epsilon)^{\kappa_{13}} K^{-2} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\kappa_{14}})$ arithmetic operations. The big- O constants are absolute and arithmetic is done using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. For simplicity, assume $\Lambda = \Lambda(K, j, \epsilon)$ is an integer. Since $\alpha_2 \neq 0$ and $y \in [0, K_1]$, $|K_1| < \infty$, the integral with respect to x in (80) is uniformly bounded in y . Thus, the order of integration in (80) does not matter. It is sufficient to compute (80) over C_0 , because over $\overline{C_0}$ we essentially obtain a conjugate sum. By the change of variable $x \rightarrow x - 1/4$, the sum (80) is then reduced to the form

$$\frac{1}{K_1^j} \sum_{k=\Lambda^2}^{K_2 - \Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^\infty \int_0^{K_1} y^j e^{2\pi i c_1 y^3 - \pi i y/2 - 2\pi i x y} e^{2\pi i \tau_k x - 2\pi i \alpha_2 x^2} dy dx \quad (81)$$

where $a_1 := a + \alpha/4$ and $\tau_k := \alpha k - \alpha_1 - \alpha_2/2$. Note $\tau_k > \alpha(\Lambda^2 - 5) > \Lambda - 1$ for $k \geq \Lambda^2$. Define

$$C_1 := \{te^{-\pi i/6} | 0 \leq t \leq 2K_1/\sqrt{3}\}, \quad C_2 := \{K_1 - it | 0 \leq t \leq K_1/\sqrt{3}\}$$

By Cauchy's Theorem we can write the integral with respect to y in (81) as the difference of integrals over C_1 and C_2 . Starting with the integral over C_1 , the sum (81) becomes

$$\frac{1}{K_1^j} \sum_{k=\Lambda^2}^{K_2 - \Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^\infty \int_0^{\frac{2K_1}{\sqrt{3}}} y^j e^{2\pi i c_1 y^3 - \pi e^{\pi i/3} y/2 - 2\pi e^{\pi i/3} x y} e^{2\pi i \tau_k x - 2\pi i \alpha_2 x^2} dy dx$$

The last integral with respect to y declines faster than $e^{-y/16}$ over $[0, 2K_1/\sqrt{3}]$. So, we can truncate with respect to y at $L = \lfloor 16\nu(K, j, \epsilon) \rfloor$, then use Taylor expansions to reduce to a sum of the form

$$\sum_{k=\Lambda^2}^{K_2 - \Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^\infty \int_0^1 y^l e^{-\pi e^{\pi i/3} (y+n)x} e^{2\pi i \tau_k x - 2\pi i \alpha_2 x^2} dy dx \quad (82)$$

where $0 \leq n < L$ and $0 \leq l = O(\nu(K, j, \epsilon))$ integers. Define

$$C_3 := \{e^{-\pi i/4} x | 0 < x < \infty\}, \quad C_4 := \{e^{\pi i/4} x | 0 < x < \infty\}$$

If α_2 is positive, then

$$\left| \int_0^T e^{-\pi e^{\pi i/3}(y+n)(T-it)} e^{2\pi i \tau_k(T-it) - 2\pi i \alpha_2(T-it)^2} dt \right| \rightarrow_{T \rightarrow \infty} 0$$

So, we can integrate x in (82) over C_3 instead of $(0, \infty)$. Similarly, if α_2 is negative, then shift to C_4 . In any case, we obtain a sum of the form

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^\infty \int_0^1 y^l e^{2\pi i e^{-sgn(\alpha_2)\pi i/4}(\tau_k - e^{-\pi i/6}(y+n))x - 2\pi|\alpha_2|x^2} dy dx \quad (83)$$

We consider both possibilities in (83): α_2 positive or α_2 negative. In the former case, the sum (83) becomes

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^\infty \int_0^1 y^l e^{2\pi(e^{\pi i/4}\tau_k - e^{\pi i/12}(y+n))x - 2\pi|\alpha_2|x^2} dy dx \quad (84)$$

For $y \in [0, 1]$ and $k \geq \Lambda^2$ we have

$$\Re\{e^{\pi i/4}\tau_k - e^{\pi i/12}(y+n)\} \geq \tau_k/2 - n - 1 \geq \Lambda/3 \quad (85)$$

In particular, in the “complementary” sum

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_{-\infty}^0 \int_0^1 y^l e^{2\pi(e^{\pi i/4}\tau_k - e^{\pi i/12}(y+n))x - 2\pi|\alpha_2|x^2} dy dx \quad (86)$$

the integrals with respect to x can be truncated at $x = -1$, which yields

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^1 \int_0^1 y^l e^{-2\pi(e^{\pi i/4}\tau_k - e^{\pi i/12}(y+n))x - 2\pi|\alpha_2|x^2} dy dx$$

Apply Taylor expansions to $e^{2\pi e^{\pi i/12}yx}$, then integrate with respect to y to reduce to the form

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^1 x^p e^{-2\pi(e^{\pi i/4}\tau_k - e^{\pi i/12}n)x - 2\pi|\alpha_2|x^2} dx \quad (87)$$

where $0 \leq p = O(\nu(K, j, \epsilon))$ an integer. Note

$$\Re\{e^{\pi i/4}\tau_k - e^{\pi i/12}n\} \geq \alpha(\Lambda^2 - 5) - 16\nu, \quad 1/\Lambda \leq \alpha_2 \leq \alpha$$

Thus, we can truncate the integral (87) with respect to x at $\min\{1, \alpha^{-1}\}$, then eliminate the quadratic factor $e^{-2\pi|\alpha_2|x^2}$ using Taylor expansions. Some further simple manipulations finally yield sums of the forms $G(K, l; a, b)$, where

$0 \leq l = O(\nu(K, j, \epsilon))$ is an integer. Such sums can be computed efficiently; see Section 1.

Now, suppose α_2 is negative in (83). Note

$$\Re\{e^{3\pi i/4}\tau_k - e^{7\pi i/12}(y+n)\} \leq -\pi\tau_k/2 + (y+n)$$

So, when α_2 is negative the real part of the coefficient of x in (83) is $\leq -\Lambda/3$ for all $y \in [0, 1]$ and $k \geq \Lambda^2$. Thus, we essentially obtain an integral of the form (86). Put together, our problem has been reduced to computing

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_{-\infty}^{\infty} \int_0^1 y^l e^{2\pi(e^{\pi i/4}\tau_k - e^{\pi i/12}(y+n))x - 2\pi\alpha_2 x^2} dy dx \quad (88)$$

where $\alpha_2 > 0$. Integrating with respect to x in (88) yields a sum of the form

$$\frac{1}{\sqrt{2\alpha_2}} \sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_2 k + 2\pi i b_1 k^2} \int_0^1 y^l e^{-2\pi \frac{e^{\pi i/3}\tau_k(y+n)}{2\alpha_2} + 2\pi \frac{e^{\pi i/6}(y+n)^2}{4\alpha_2}} dy \quad (89)$$

where $a_2 = O(\alpha^2/\alpha_2)$ and $b_1 = O(\alpha^2/\alpha_2)$ are real numbers. Of course, both a_2 and b_1 can be reduced mod 1. We point out their “raw magnitude” to show that they can be expressed in $O(\nu(K, j, \epsilon)^2)$ bits throughout, as do all other numbers in this paper. The integrand in (89) declines very rapidly over $[0, 1]$. In fact, it is completely negligible if $n > 0$. So, we can assume $n = 0$. We truncate the integral with respect to y at α_2/α , then eliminate the quadratic factor $e^{\pi e^{\pi i/6}y^2/(2\alpha_2)}$ using Taylor expansions. Finally, some further simple manipulations reduce the problem to evaluating sums of the form $G(K, l; a, b)$, where $0 \leq l = O(\nu(K, j, \epsilon))$ is an integer. Such sums can be evaluated efficiently; see Section 1.

Having shown how to deal with the integral over C_1 , we now move on to the contour C_2 . Here we reduce to sums of the form

$$\frac{1}{K_1^j} \sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{2\pi i a_1 k + 2\pi i b k^2} \int_0^{\infty} \int_0^{\frac{K_1}{\sqrt{3}}} (K_1 - iy)^j \times \quad (90)$$

$$e^{6\pi c_1 K_1^2 y - 6\pi i c_1 K_1 y^2 - 2\pi c_1 y^3 - \pi y/2 - 2\pi x y} e^{-2\pi i \tilde{\tau}_k x - 2\pi i \alpha_2 x^2} dy dx$$

where $\tilde{\tau}_k := K_1 - \tau_k$. Observe that $\tilde{\tau}_k > \alpha(\Lambda^2 - 5) > \Lambda - 1$ for $k \leq K_2 - \Lambda^2$. Truncate the integral with respect to y in (90) at $L = \lfloor 16\nu(K, j, \epsilon) \rfloor$, then use Taylor expansions to reduce to sums of the form

$$\sum_{k=\Lambda^2}^{K_2-\Lambda^2} e^{\pi i a_1 k + 2\pi i b k^2} \int_0^{\infty} \int_0^1 y^l e^{-2\pi x(y+n)} e^{-2\pi i \tilde{\tau}_k x - 2\pi i \alpha_2 x^2} dy dx \quad (91)$$

where $0 \leq n < L$ and $0 \leq l = O(\nu(K, j, \epsilon))$ are integers. Finally, (91) is treated in a completely analogous way to (82). \square

Lemma 5.7. *There exist absolute constants κ_{15} and κ_{16} such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any positive integer $K > \Lambda(K, j, \epsilon)$, any positive integer $N \leq K$, any real c satisfying $1/\Lambda(K, j, \epsilon) \leq |cN^2| \leq 2/\Lambda(K, j, \epsilon)$, any real number α and positive integer M satisfying $\alpha \in [1/\Lambda(K, j, \epsilon), N/M]$ and $\Lambda(K, j, \epsilon)^2 \leq M \leq N/\alpha$, any $\alpha_1 \in [0, 4\alpha]$, any real α_2 satisfying $|\alpha_2| \in [1/K^2, \alpha]$, any integer $k \in [T, M - T]$ where $T := \lfloor \Lambda(K, j, \epsilon)^2 \rfloor$, and for*

$$\begin{aligned} f_k(y) &= f_k(y, c, \alpha, \alpha_1, \alpha_2) := cy^3 - \frac{y^2}{4\alpha_2} + \frac{\alpha k - \alpha_1}{2\alpha_2}y - \frac{(\alpha k - \alpha_1)^2}{4\alpha_2} \\ y_k &= y_k(c, \alpha, \alpha_1, \alpha_2) := \frac{1}{12c\alpha_2} \left(1 - \sqrt{1 - 24c\alpha_2(\alpha k - \alpha_1)} \right) \\ h_k(y) &= h_k(y, c, \alpha_2) := cy^3 + \left(3cy_k - \frac{1}{4\alpha_2} \right) y^2, \end{aligned}$$

the function

$$I_{k,j} = I_{k,j}(c, \alpha, \alpha_1, \alpha_2, N) := \frac{1}{\sqrt{2|\alpha_2|}N^j} \int_0^N y^j e^{2\pi i h_k(y-y_k)} dy$$

can be written in the form

$$\begin{aligned} I_{k,j} &= \sum_{l=0}^L z_l \frac{k^l}{M^l} + e^{2\pi i a_1 k + 2\pi i b_1 k^2 - 2\pi i f_k(y_k)} \sum_{l=0}^L w_l \frac{k^l}{M^l} \\ &\quad + e^{2\pi i a_1 k + 2\pi i b_1 k^2 - 2\pi i f_k(y_k)} \sum_{l=0}^L \tilde{w}_l \frac{1}{k^l} + e^{2\pi i a_2 k + 2\pi i b_2 k^2 - 2\pi i f_k(y_k)} \sum_{l=0}^L v_l \frac{k^l}{M^l} \\ &\quad + e^{2\pi i a_2 k + 2\pi i b_2 k^2 - 2\pi i f_k(y_k)} \sum_{l=0}^L \tilde{v}_l \frac{1}{k^l} + O(\nu(K, j, \epsilon)^{\kappa_{15}} K^{-2} \epsilon) \end{aligned}$$

where $L = O(\nu(K, j, \epsilon)^{\kappa_{16}})$, the numbers a_1, b_1, a_2 , and b_2 are in \mathbb{R}/\mathbb{Z} and are independent of k , the coefficients z_l, w_l , and v_l are independent of k and are bounded by $O(1)$, and the coefficients \tilde{w}_l and \tilde{v}_l are independent of k and are bounded by $O(\Lambda(K, j, \epsilon)^l)$. The absolute constant κ_{16} can be taken to be equal to 6. Finally, big- O constants are absolute and arithmetic is done using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. First, observe

$$\begin{aligned} y_k &= \sum_{l=0}^{\infty} u_l (24c\alpha_2)^{l-1} (\alpha k - \alpha_1)^l \\ &= \sum_{s=0}^{\infty} \alpha^s k^s (24c\alpha_2)^{s-1} \sum_{l=0}^{\infty} u_{l+s} \binom{l+s}{s} (24c\alpha_1\alpha_2)^l \end{aligned} \tag{92}$$

where $u_0 = 0$, $u_1 := 1$, and $|u_l| \leq 1$ for $l > 1$. So,

$$\begin{aligned} y_k &= \sum_{s=0}^{\infty} g_s \alpha^s k^s, \quad \text{where} \\ g_s &:= u_s (24c\alpha_2)^{s-1} + \alpha_1 (24c\alpha_2)^s \sum_{l=0}^{\infty} u_{l+s+1} \binom{l+s+1}{s} (24c\alpha_1\alpha_2)^l \end{aligned} \quad (93)$$

In particular, $|g_0| \leq 2\alpha_1$, $g_1 = 1 + O(M^{-2})$, and for $s \geq 0$ we have

$$\begin{aligned} |g_{s+1}| &\leq |24c\alpha_2|^s + 2\alpha_1 |48c\alpha_2|^{s+1} \sum_{l=0}^{\infty} |48c\alpha_1\alpha_2|^l \\ &\leq \frac{1}{(MN)^s} + \frac{1}{M^2(MN)^s} \leq \frac{2}{(MN)^s} \end{aligned} \quad (94)$$

Therefore, $y_k = \alpha k + \Delta_1$, where $|\Delta_1| < 20\alpha$. So, for $T \leq k \leq M - T$, we have $\alpha(T - 20) < y_k < N - \alpha(T - 20)$. We split $I_{k,j}$ into the two integrals

$$I_{k,j}^1 := \int_0^{N-y_k} \frac{(y_k + y)^j}{\sqrt{2|\alpha_2|} N^j} e^{2\pi i h_k(y)} dy, \quad I_{k,j}^2 := \int_0^{y_k} \frac{(y_k - y)^j}{\sqrt{2|\alpha_2|} N^j} e^{2\pi i h_k(-y)} dy$$

We show how to evaluate $I_{k,j}^2$ first. Conjugating if necessary, we may assume the parameter α_2 that appears in the definition of $h_k(-y)$ is negative (note conjugation switches the sign of c also). By Cauchy's Theorem

$$I_{k,j}^2 = \int_0^{\sqrt{2}y_k} \frac{e^{\pi i/4} (y_k - e^{\pi i/4} y)^j}{\sqrt{2|\alpha_2|} N^j} e^{2\pi i h_k(-e^{\pi i/4} y)} dy + \int_0^{y_k} \frac{(-i)^{j+1} y^j}{\sqrt{2|\alpha_2|} N^j} e^{2\pi i h_k(-(y_k + iy))} dy$$

Simple manipulations then yield

$$I_{k,j}^2 = e^{\pi i/4} I_{k,j}^{2,1} + (-i)^{j+1} e^{2\pi i g(y_k)} I_{k,j}^{2,2} \quad (95)$$

where

$$\begin{aligned} g(y) &:= 2cy^3 - \frac{y^2}{4\alpha_2} \\ I_{k,l}^{2,1} &:= \int_0^{\sqrt{2}y_k} \frac{(y_k - e^{\pi i/4} y)^j}{\sqrt{2|\alpha_2|} N^j} e^{-2\pi \left(3cy_k + \frac{1}{4|\alpha_2|}\right) y^2 + 2\pi e^{\pi i/4} cy^3} dy \\ I_{k,l}^{2,2} &:= \int_0^{y_k} \frac{y^j}{\sqrt{2|\alpha_2|} N^j} e^{-2\pi \left(3\pi cy_k + \frac{1}{2|\alpha_2|}\right) y_k y - \frac{2\pi i}{4|\alpha_2|} y^2 - 2\pi cy^3} dy \end{aligned}$$

Note that if $I_{k,j}^2$ was conjugated in order to ensure α_2 was negative, then once we reach (95), we would conjugate back. So, the coefficients α_2 and c that appear in the definition of $g(y)$ above are the same as the coefficients that occur in the statement of the Lemma. Now,

$$3cy_k + \frac{1}{4|\alpha_2|} = \frac{\sqrt{1 + 24c|\alpha_2|(\alpha k - \alpha_1)}}{4|\alpha_2|} \geq \frac{1}{8|\alpha_2|}$$

Also, the condition $k \geq T$ ensures that $\sqrt{2}y_k > \sqrt{2|\alpha_2|}\lfloor\sqrt{T}\rfloor$. Let us show how to compute $I_{k,j}^{2,1}$ first. Truncate $I_{k,l}^{2,1}$ at $\sqrt{2|\alpha_2|}\lfloor\sqrt{T}\rfloor$, then eliminate the factors $e^{2\pi e^{\pi i/4}cy^3}$ and $e^{-6\pi cy_k y^2}$ using Taylor expansions. We reduce to

$$(6\pi cy_k)^v (2\pi c)^d \int_0^{\sqrt{2|\alpha_2|}\lfloor\sqrt{T}\rfloor} \frac{(y_k - e^{\pi i/4}y)^j}{\sqrt{2|\alpha_2|}N^j} y^{2v+3d} e^{-\frac{\pi y^2}{2|\alpha_2|}} dy$$

where $0 \leq d = O(\nu(K, j, \epsilon))$, $0 \leq v = O(\nu(K, j, \epsilon))$ are integers. Next, make a change of variable $y/\sqrt{2|\alpha_2|} \rightarrow y$. This simplifies the problem to evaluating

$$(6\pi c|\alpha_2|y_k)^v (2\pi c|\alpha_2|^{3/2})^d \int_0^{\lfloor\sqrt{T}\rfloor} \frac{(y_k - e^{\pi i/4}\sqrt{2|\alpha_2|}y)^j}{N^j} y^{2v+3d} e^{-\pi y^2} dy \quad (96)$$

Next, expand the factor $(y_k - e^{\pi i/4}\sqrt{2|\alpha_2|}y)^j$ in (96). We reduce to expressions of the form $(y_k^{l+v}/N^{l+v})\tilde{I}_{j,l,v,d}$ where $0 \leq l \leq j$ is an integer and

$$\tilde{I}_{j,l,v,d} := (6\pi cN|\alpha_2|)^v (2\pi c|\alpha_2|^{3/2})^d \binom{j}{l} \frac{(2|\alpha_2|)^{(j-l)/2}}{N^{j-l}} \int_0^{\lfloor\sqrt{T}\rfloor} y^{j-l+2v+3d} e^{-\pi y^2} dy$$

Note that $\tilde{I}_{j,l,v,d}$ is bounded by $O(1)$, is independent of k , and can be easily computed with error bounded by $O(K^{-2}\epsilon)$ using $O(\Lambda(K, j, \epsilon))$ operations. Therefore, we may turn our attention to the term y_k^{l+v}/N^{l+v} . By (92), (93), and (94), we have

$$\frac{y_k}{N} = \sum_{s=0}^{\infty} \frac{g_s \alpha^s k^s}{N} = h_0 + \sum_{s=1}^R h_s \frac{k^s}{M^s} + O(K^{-3}\epsilon)$$

where $R := \lfloor 3\nu(K, j, \epsilon) \rfloor$, $|h_0| \leq 8/M$, $|h_1| \leq 1 + O(M^{-2})$, and for $s \geq 2$

$$|h_s| = \left| \frac{g_s \alpha^s M^s}{N} \right| \leq \frac{2N^s}{N(MN)^{s-1}} \leq \frac{2}{M^{s-1}}$$

Let $S := \sum_{s=2}^R h_s \frac{k^s}{M^s}$. For any integer $0 \leq q = O(\nu(K, j, \epsilon))$ and any number $0 \leq \delta = O(K^{-3}\epsilon)$ we have

$$\begin{aligned} ((\delta + h_0 + h_1 k/M) + S)^q &= \sum_{p=0}^q \binom{q}{p} (\delta + h_0 + h_1 k/M)^{q-p} S^p \\ &= \sum_{p=0}^q (\delta + h_0 + h_1 k/M)^{q-p} \sum_{s=2p}^{Rp} \tilde{h}_{s,p} \frac{k^s}{M^s} \end{aligned}$$

where $\tilde{h}_{s,p} = O(R^p/M^{s/2}) = O(1/\Lambda(K, j, \epsilon))$ for $s \geq 2p$. So,

$$((\delta + h_0 + h_1 k/M) + S)^q = \sum_{p=0}^q (\delta + h_0 + h_1 k/M)^{q-p} \sum_{s=2p}^{Rp} \tilde{h}_{s,p} \frac{k^s}{M^s} + O(K^{-2}\epsilon)$$

Therefore, we can expand $(y_k/N)^{l+v}$ as a power series in k/M of length $O(\nu(K, j, \epsilon)^2)$. The truncation error is $O(K^{-2}\epsilon)$ and coefficients of the power series are bounded by $O(1/\nu(K, j, \epsilon))$.

Having shown how to compute $I_{k,j}^{2,1}$, we now deal with $I_{k,j}^{2,2}$. Before doing so, let us first observe

$$g(y) + f_k(y) = yf'_k(y) - \frac{(\alpha k - \alpha_1)^2}{4\alpha_2} \Rightarrow g(y_k) = -f_k(y_k) - \frac{(\alpha k - \alpha_1)^2}{4\alpha_2}$$

Thus, the factor $e^{2\pi i g(y_k)}$ in (95) can be replaced with a factor of the form $e^{2\pi i \alpha_1 k + 2\pi i b_1 k^2 - 2\pi i f_k(y_k)}$. We truncate $I_{k,j}^{2,2}$ at $\sqrt{2|\alpha_2|}$, then make the change of variable $y/\sqrt{2|\alpha_2|} \rightarrow y$. We reduce to an integral of the form

$$\int_0^1 \frac{(2|\alpha_2|)^{j/2} y^j}{N^j} e^{-6\pi c \sqrt{2|\alpha_2|} y_k y - \frac{2\pi}{\sqrt{2|\alpha_2|}} y_k y - \pi i y^2 - 2\pi c (2|\alpha_2|)^{3/2} y^3} dy$$

Next, eliminate the factors $e^{-\pi i y^2}$ and $e^{-2\pi c (2|\alpha_2|)^{3/2} y^3}$ using Taylor expansions. This yields integrals of the form

$$\frac{(2|\alpha_2|)^{j/2}}{N^j} \int_0^1 y^{j+u} e^{-2\pi \left(3c\sqrt{2|\alpha_2|} + \frac{1}{\sqrt{2|\alpha_2|}}\right) y_k y} dy \quad (97)$$

where $0 \leq u = O(\nu(K, j, \epsilon))$ is an integer. Integrate with respect to y in (97) to obtain terms of the form

$$\frac{(2|\alpha_2|)^{j/2} (j+u)!}{N^j (j+u+1-r)!} (2\pi \tilde{\alpha})^{-r} y_k^{-r}, \quad \tilde{\alpha} := 3c\sqrt{2|\alpha_2|} + \frac{1}{\sqrt{2|\alpha_2|}}$$

where $1 \leq r \leq j+u+1$ is an integer. But

$$\begin{aligned} y_k^{-1} &= \frac{1 + \sqrt{1 - 24c\alpha_2(\alpha k - \alpha_1)}}{2(\alpha k - \alpha_1)} \\ &= \frac{1}{\alpha k - \alpha_1} + \frac{1}{\alpha k - \alpha_1} \sum_{s=1}^{\infty} \tilde{g}_s \frac{(\alpha k - \alpha_1)^s}{N^s} \end{aligned}$$

where $\tilde{g}_s \leq (24c\alpha_2 N)^s \leq M^{-s}$. Therefore

$$\begin{aligned} \frac{(2\pi \tilde{\alpha})^{-r} (2|\alpha_2|)^{j/2} (j+u)!}{N^j (j+u+1-r)!} y_k^{-r} &= \frac{\tilde{C}_{j,r,u}}{(\alpha k - \alpha_1)^r} \sum_{w=0}^r \binom{r}{w} \left(\sum_{s=1}^{\infty} \tilde{g}_s \frac{(\alpha k - \alpha_1)^s}{N^s} \right)^w \\ &= \frac{\tilde{C}_{j,r,u}}{(\alpha k - \alpha_1)^r} \sum_{w=0}^r \sum_{s=w}^{\infty} \tilde{f}_{s,w} \frac{(\alpha k - \alpha_1)^s}{N^s} \end{aligned}$$

where $|\tilde{f}_{s,w}| \leq r^w s^w M^{-s} = O(M^{-s/2})$ for $s \geq w$, and where the constants $\tilde{C}_{j,r,u} = O((j+u)^r (2\alpha)^{r/2})$. So,

$$\frac{(2\pi \tilde{\alpha})^{-r} (2|\alpha_2|)^{j/2} (j+u)!}{N^j (j+u+1-r)!} y_k^{-r} = \frac{\tilde{C}_{j,r,u}}{(\alpha k - \alpha_1)^r} \sum_{w=0}^r \sum_{s=w}^R \tilde{f}_{s,w} \frac{(\alpha k - \alpha_1)^s}{N^s} + O(K^{-2}\epsilon)$$

where $R = \lfloor 3\nu(K, j, \epsilon) \rfloor$. Some further simple manipulations give the result in its final form.

As for $I_{k,j}^1$, its computation mirrors that of $I_{k,j}^2$. The details of treatment differ from those of $I_{k,j}^2$ mainly in so far as we appeal to the bound $k \leq M - T$ rather than $k \geq T$. \square

Lemma 5.8. *For any integers n and R satisfying $0 < n < R$ the quantity*

$$\begin{aligned} & 2^{(\lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil + 1)(R/n+3 - \lceil R/2n+3/2 \rceil)n} \times \\ & \prod_{s=0}^{\lceil R/2n+3/2 \rceil - 1} 2^{sn} \prod_{s=0}^{\lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil} 2^{-sn} \end{aligned} \quad (98)$$

is dominated by $2^{n(R/2n+3/2)^2}$.

Proof. For $x \in \mathbb{R}$, let $\bar{x} := x \pmod{1}$ such that $\bar{x} \in [0, 1)$. It is easy to verify

$$2\lceil x \rceil - \lfloor 2x \rfloor = 2\delta_{\bar{x} \in (0, 1/2)} + \delta_{\bar{x} \in [1/2, 1)}$$

So

$$\lceil R/2n+3/2 \rceil - 1 = \begin{cases} \lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil + 1 & \text{if } \overline{R/2n+3/2} \in (0, 1/2) \\ \lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil & \text{if } \overline{R/2n+3/2} \in [1/2, 1) \\ \lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil - 1 & \text{if } \overline{R/2n+3/2} = 0 \end{cases}$$

Suppose $\overline{R/2n+3/2} \in (0, 1/2)$. Define $\lambda := \{R/2n+3/2\}$. By hypothesis, $\lambda > 0$; in particular, $\lceil R/2n+3/2 \rceil = R/2n+3/2 - \lambda + 1$. So, (98) is equal to

$$\begin{aligned} & 2^{(\lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil + 1)(R/n+3 - \lceil R/2n+3/2 \rceil)n} 2^{(\lceil R/2n+3/2 \rceil - 1)n} \\ &= 2^{(R/2n+3/2 - \lambda)(R/n+4 - (R/2n+3/2 - \lambda + 1))n} \\ &= 2^{n(R/2n+3/2)^2 - n\lambda^2} \end{aligned}$$

Now, suppose $\overline{R/2n+3/2} \in [1/2, 1)$. Define $\tilde{\lambda} := 1 - \{R/2n+3/2\}$. By hypothesis $\tilde{\lambda} > 0$; so $\lceil R/2n+3/2 \rceil = R/2n+3/2 + \tilde{\lambda}$. Then, (98) is equal to

$$\begin{aligned} & 2^{(\lfloor R/n+3 \rfloor - \lceil R/2n+3/2 \rceil + 1)(R/n+3 - \lceil R/2n+3/2 \rceil)n} \\ &= 2^{(R/2n+3/2 + \tilde{\lambda})(R/n+3 - (R/2n+3/2 + \tilde{\lambda}))n} \\ &= 2^{n(R/2n+3/2)^2 - n\tilde{\lambda}^2} \end{aligned}$$

Finally, suppose $\overline{R/2n+3/2} = 0$. Then, $\overline{R/n+3} = 0$ and (98) is equal to $2^{n(R/2n+3/2)^2}$. \square

Lemma 5.9. *There exist absolute constants κ_{17} and κ_{18} such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any integer K_1 satisfying $\Lambda(K, j, \epsilon) < K_1 < K$, any integer K_2 satisfying $K_1 < K_2 \leq K$, any $\alpha \in [0, 1]$, any $\alpha_1 \in [0, 4\alpha]$, any real α_2 satisfying $0 \leq |\alpha_2| \leq \alpha$, any real c_1 satisfying $|c_1 K_1^2| < 1/48$, and any a, b satisfying*

$$\begin{aligned} (a + \alpha x, b) &\in (0, 2) \times [0, 1/4] && \text{for all } x \in [-1/4, 1/4] \\ \lceil a + \alpha x \rceil &\geq \lfloor a + \alpha x + 2bK_2 \rfloor && \text{for some } x \in [-1/4, 1/4], \end{aligned} \quad (99)$$

the sum

$$\frac{1}{K_1^j} \int_{-1/4}^{1/4} \int_0^{K_1} y^j e^{2\pi i c_1 y^3 - 2\pi i x y} e^{-2\pi i \alpha_1 x - 2\pi i \alpha_2 x^2} F(K_2; a + \alpha x, b) dy dx \quad (100)$$

can be computed with error bounded by $O(\nu(K, j, \epsilon)^{\kappa_{17}} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\kappa_{18}})$ arithmetic operations. The big- O constants are absolute and arithmetic is done using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. Since $\alpha \leq 1$, then $\alpha_1 \leq 4$ and $|\alpha_2| \leq 1$. So, (100) may be reduced via Taylor expansions to the form

$$\frac{1}{K_1^j} \int_{-1/4}^{1/4} \int_0^{K_1} y^j x^p e^{2\pi i c_1 y^3 - 2\pi i x y} F(K_2; a + \alpha x, b) dy dx \quad (101)$$

where $0 \leq p = O(\nu(K, j, \epsilon))$ is an integer. The hypothesis (99) implies that $0 \leq a + \alpha x + 2bK_2 < 4$ for all $x \in [-1/4, 1/4]$, so $2bK_2 < 4$. Without loss of generality, we may assume that K_2 is a multiple of 32. Subdivide $F(K_2; a + \alpha x, b)$ into 32 consecutive subsums to reduce the problem to evaluating sums of the form

$$\frac{1}{K_1^j} \int_{-1/4}^{1/4} \int_0^{K_1} y^j x^p e^{2\pi i c_1 y^3 - 2\pi i x y + 2\pi i \tilde{K}_m x} F(K_3; a_m + \alpha x, b) dy dx \quad (102)$$

where $0 \leq m < 32$ an integer, $a_m := a + bmK_2/16$, $\tilde{K}_m := \alpha mK_2/32$, and $K_3 := K_2/32 - 1$. We can normalize a_m so that $a_m + \alpha x \in [-3/4, 3/4]$ for all $x \in [-1/4, 1/4]$. In particular, we can assume $|a_m| + |\alpha x| + 2bK_3 < 7/8$ for all $x \in [-1/4, 1/4]$. Given this, we can apply Euler-Maclaurin summation to the sum $F(K_3; a_1 + \alpha x, b)$ in (102). The problem is essentially reduced to computing the integral

$$\frac{1}{K_1^j} \int_{-1/4}^{1/4} \int_0^{K_1} \int_0^{K_3} y^j x^p e^{2\pi i c_1 y^3 - 2\pi i x y + 2\pi i \tilde{K}_m x} e^{2\pi i \alpha x z + 2\pi i a_m z + 2\pi i b z^2} dz dy dx$$

Finally, Cauchy's Theorem as well as saddle-point techniques similar to the ones we carried out before allow to reduce the last integral to integrals of the type handled by Lemma 5.5. \square

Lemma 5.10. *There exist absolute constants κ_{19} and κ_{20} such that for any $\epsilon \in (0, e^{-1})$, any integer $j \geq 0$, any integer $K > \Lambda(K, j, \epsilon)$, any integer K_1 satisfying $0 \leq K_1 \leq K$, any $a, b \in \mathbb{R}/\mathbb{Z}$, any integer $0 \leq l = O(\nu(K, j, \epsilon))$, any $\alpha \in [-1, 1]$, and any $\beta \geq 0$, the sum*

$$\int_0^1 x^l e^{2\pi i K_1 x} F(K; a + \alpha x + i\lambda x, b) dx \quad (103)$$

can be computed with error bounded by $O(\nu(K, j, \epsilon)^{\kappa_{19}} \epsilon)$ using $O(\nu(K, j, \epsilon)^{\kappa_{20}})$ arithmetic operations. The big- O constants are absolute and arithmetic is done using $O(\nu(K, j, \epsilon)^2)$ bits.

Proof. Let us first assume that $\lambda = 0$. We also assume that $|\alpha| > 1/K$, otherwise the sum reduces to the form $F(K; a, b)$, which can be computed efficiently; see Section 1.

Let M denote the set of integers $\{k \in [0, K] \mid |K_1 + \alpha k| \leq 2(l+1)\}$. If M is not empty, then it has the form $[s, t]$, for some integer $s, t \in [0, K]$. So, consider the sum

$$\sum_{k=s}^t e^{2\pi i a k + 2\pi i b k^2} \int_0^1 x^l e^{2\pi i (K_1 + \alpha k)x} dx \quad (104)$$

Make the change of variable $x \rightarrow 2(l+1)x$ in (104). Then, subdivide the resulting integral into $2(l+1)$ consecutive subintegrals of the form

$$\sum_{k=s}^t e^{2\pi i a k + 2\pi i b k^2} \int_0^1 \frac{(x+n)^l}{(2(l+1))^l} e^{2\pi i (K_1 + \alpha k)x/(2(l+1))} dx \quad (105)$$

where $0 \leq n < 2(l+1)$ an integer. Next, apply Taylor expansions to the term $e^{2\pi i (K_1 + \alpha k)x/(2(l+1))}$ and expand $(x+n)^l$ to reduce (105) to a sum of the form

$$\frac{1}{q! (2(l+1))^q} \sum_{k=0}^{t-s} (\alpha k + \alpha s + K_1)^q e^{2\pi i \bar{a} k + 2\pi i b k^2} \int_0^1 x^p dx \quad (106)$$

where $0 \leq p = O(\nu(K, j, \epsilon))$ and $0 \leq q = O(\nu(K, j, \epsilon))$ are integers. Note by construction, $|\alpha s + K_1| \leq 2(l+1)$ and $|\alpha(t-s)| \leq 4(l+1)$. We may assume $t-s > 0$. Now, expand the term $(\alpha k + \alpha s + K_1)^q$ to obtain sums of the form

$$\binom{q}{d} \frac{(\alpha s + K_1)^{q-d} \alpha^d}{q! (2(l+1))^q} \sum_{k=0}^{t-s} k^d e^{2\pi i \bar{a} k + 2\pi i b k^2} \int_0^1 x^p dx \quad (107)$$

where $0 \leq d \leq q$ an integer. Integrate directly with respect to x in (107), then note that the constant in front of the sum (107) has magnitude less than $5^q |\alpha|^d / (q! (l+1)^d) \leq e^5 / (t-s)^d$. So, the resulting quadratic sums can be handled by Theorem 2.1; see Section 1.

It remains to consider the (possible empty) intervals $[0, s-1]$ and $[t+1, K]$. Over these intervals, directly integrate with respect to x in (103). We obtain sums of the form

$$\frac{l!}{(l+1-v)!} \sum_{k=u}^r \frac{e^{2\pi i \bar{a} k + 2\pi i b k^2}}{(K_1 + \alpha k)^v} \quad (108)$$

where $1 \leq v \leq l+1$ is an integer and $|K_1 + \alpha k| > 2(l+1)$ for all integers $k \in [u, r]$. It is then not too hard to see that the sums (108) can be subdivided into $O(\log K)$ sums of the form

$$\frac{l!}{(l+1-v)!} \sum_{k=N}^{N+\Delta} \frac{e^{2\pi i \bar{a} k + 2\pi i b k^2}}{(K_1 + \alpha k)^v} \quad (109)$$

where $N \in [u, r]$ an integer, $\Delta \in [0, N]$ an integer, $|K_1 + \alpha N| > 2(l+1)$, $|\Delta/(K_1 + \alpha N)| \leq 1/2$, plus a remainder sum of length $O(\log K)$ that can be

computed directly. In particular, the sum (109) can be reduced via Taylor expansions to sums of the type handled by Theorem 2.1; see Section 1.

Finally, the treatment of (103) when $\lambda > 0$ is analogous. \square

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